RELATING PLANE TRANSFORMATIONS WITH STEREOGRAPHIC PROJECTION

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ABSTRACT

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Stereographic projection is a type of transformation mapping points on a sphere of dimension $n + 1$ onto a plane of dimension n. It has properties such as continuity and preserving of certain angles, which lend to exploring properties of the plane in relation to the sphere. Here stereographic projection is used as a method of drawing relationships between different two-dimensional spaces. Each of the complex plane, the split-complex plane, and the hyperbolic plane are examined in their relation to the Euclidean plane. First, Möbius transformations are considered on the complex plane. It is shown that every Möbius transformation can be represented via a movement of the sphere between two stereographic projections. Second, similar transformations are represented on the split-complex plane. Laguerre transformations, analogous to Möbius transformations, explored with respect to split-complex numbers. Finally, a model of the hyperbolic plane is constructed using a pair of lateral projection and central projection.

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CHAPTER I

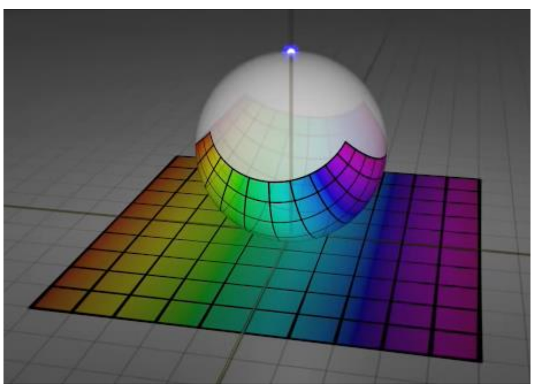
STEREOGRAPHIC PROJECTION

Introduction

Stereographic projection is a transformation mapping points on a sphere to points on a plane. More specifically: let S be a sphere and P a plane with orthogonal vector $\vec{n}$. Let N be the point on S in the direction of $\vec{n}$ from the center of the sphere. Then the stereographic projection from S onto P is the function $s : S \rightarrow P$ such that $N, X \in S \backslash N$, and $s(X) \in P$ are collinear. Figure 1 shows the map of a partial coloration of a sphere under stereographic projection.

Figure 1

Stereographic Projection



Note. The sphere shown sits atop a plane with a colored grid. Stereographic projection is represented by a coloration on the sphere corresponding to the coloration of the grid. From "Möbius Transformations Revealed," video by D. N. Arnold and J. Rogness, 2007. Copyright 2007 by the University of Minnesota.

Stereographic projection has the properties of being a continuous mapping and a conformal mapping, i.e., angles between curves are preserved.

Algebraic Representation of $s(X)$

It will become necessary to have a general formulation for where to find the image of a point under stereographic projection. Consider orienting the plane P in a three-dimensional coordinate system such that P lies on the XY plane, i.e. P is the plane $z = 0$ with the normal vector $\vec{n} = \langle 0,0,1 \rangle$. Let S be a unit sphere with center (h, k, l), characterized by the equation $(x - h)2 + (y - k)2 + (z - l)2 = 1$. Then $N = (h, k, l + 1)$ and the image of a point $X_0 = (x_0, y_0, z_0)$ under the stereographic projection $s(X)$ is found by intersecting the line $N\ X_0$ with the plane $z = 0$.

The line NX_0 can be parameterized as follows:

$$x(t) = (x_0 - h)t + h$$

$$y(t) = (y_0 - k)t + k$$

$$z(t) = (z_0 - l - 1)t + l + 1$$

Intersecting this line with the plane $z = 0$ yields:

$$0 = (z_0 - l - 1)t + l + 1$$

$$t = \frac{l + 1}{l + 1 - z_0}$$

$$x = \frac{(l + 1)(x_0 - h)}{l + 1 - z_0} + h$$

$$y = \frac{(l + 1)(y_0 - k)}{l + 1 - z_0} + k$$

It follows that:

$$s(x_0, y_0, z_0) = \left(\frac{(l+1)(x_0 - h)}{l+1-z_0} + h, \frac{(l+1)(y_0 - k)}{l+1-z_0} + k, 0\right)$$

Compactification of $\mathbb{R}^2$

The previous definition of stereographic projection does not define an image for the point N under this function. One might observe the points close to N on the sphere are mapped to points very far from the origin on the plane. Define a point ∞ such that $\lim_{X \to N} s(X) = \infty$. Then as $s(N)$ is not yet defined, letting $s(N) = \infty$ preserves the continuity of the stereographic projection.

This new point ∞ and the stereographic projection of the sphere onto the plane yield a single-point compactification of $\mathbb{R}^2$ and any other two-dimensional plane as a direct result. Compactness is defined as follows: a space X is said to be $Compact$ if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X. With a bit of experimentation on open sets, one can generally find an open covering without a finite subcovering, showing that open sets are not compact. For example, consider the open interval $(0,1)$ in $\mathbb{R}$. This interval is covered by the infinite union of sets: $\bigcup_{k=2}^{\infty} \left(\frac{1}{k}, 1\right)$. Figure 2 shows the union of the first eight sets, while Figure 3 shows the union of the first 25 sets.

Figure 2

Partial Covering on (0,1)

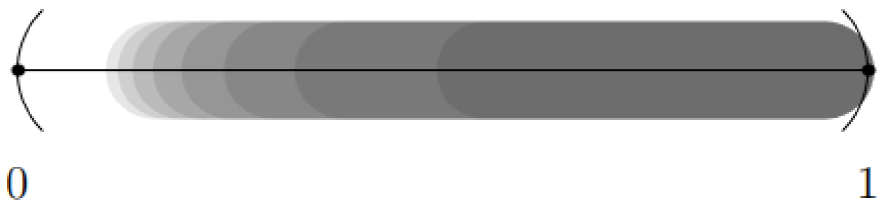


Figure 3

A Near-Complete Covering on (0,1)

For any real number n between 0 and 1, there exists some positive integer k such that $n > \frac{1}{k}$, so this is a full covering of $(0, 1)$. However, if one attempts to take any finite subcover one finds there exists a positive n close enough to 0 that is not included in the subcover. Hence, this open interval is not compact.

Similar considerations can be performed on any open interval (a, b), using the covering union of sets: $\bigcup_{k=1}^{\infty}\left(\frac{1}{k} + a, b\right)$ for which there is no finite subcover, as any finite subcover will leave a gap near a. Hence, any open interval appears to be non-compact.

In general, having an open border on a set gives the ability to have an open covering made of sets which come infinitely close to the border as above. One might find it intuitive that compact sets in $\mathbb{R}$ are exactly those which are closed and bounded.

For proof, let $\cup A_i$ be an open covering of $[a,b] \subset \mathbb{R}$. Take some $x \in [a,b)$, then there exists A_i such that $x \in A_i$. As A_i is open, by definition it must contain an open ball around x, i.e., $\exists c \in [a,b] \ni [x,c) \subseteq A_i$. Then for $y \in [x,c)$ there exists a closed interval $[x,y] \subseteq [a,b]$, which is covered finitely, by a single open set A_i.

Since a finitely coverable interval exists starting at any $x \in [a,b)$, one might consider such intervals that begin at a. Let C be the set of all points $y \in [a,b]$ such that $[a,y]$ is finitely coverable. Suppose $c < b$ is the least upper bound of C, i.e., there exists a finite covering of $[a,c)$. Then among the covering sets there must be a set A_k such that $c \in A_k$ and therefore an element $d \in A_k$ with $[d,c] \subset A_k$. As $[a,d] \subset [a,c)$ it follows that $[a,d]$ is finitely coverable. Then $[a,d] \cup [d,c]$ is finitely coverable as well, as a union of two finitely coverable sets.

It is now known that $[a,c]$ is finitely coverable. Point c was previously defined as the least upper bound among finitely coverable sets with lower bound a. However, note that from the first assertion, it was found that for any $x \in [a,b)$ there exists $y \in (x,b]$ such that $[x,y]$ is finitely coverable. Applying this idea to c, it means there is some $y > c$ such that $[c,y]$ is finitely coverable. Then $[a,c] \cup [c,y] = [a,y]$ is finitely coverable, contradicting the idea that a least upper bound exists on our finitely coverable intervals. It follows that for any $y > a$, the interval $[a,y]$ is finitely coverable. Thus $[a,b]$ has a finite subcover in $\cup A_i$ and is compact.

This idea extends to $\mathbb{R}^n$, resulting in what is known as the Heine-Borel Theorem, as stated by Munkres (2000): A subset S of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Returning to the stereographic projection between the sphere and the plane, note that the sphere S is a closed and bounded set of points in $\mathbb{R}^3$. This implies that the sphere represents a compact set of points. The image of a compact set under any continuous function is also compact, as the image of a finite subcover will cover the image of the compact set. Thus, the image of the sphere S under the stereographic projection $s(X)$ is a compact set. The image is exactly $P \cup \infty$, $so\ P \cup \infty$ is compact. Let $\hat{P} = P \cup \infty$, we call $\hat{P}$ the single-point compactification of P.

CHAPTER II

THE COMPLEX PLANE

Introduction

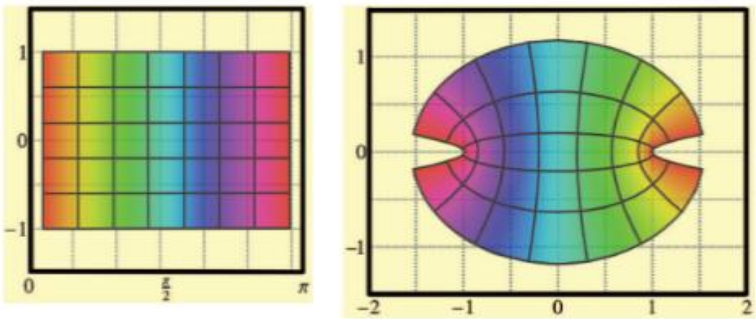
The complex plane $\mathbb{C}$ is defined by the set of points $x + yi$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. The complex plane is homeomorphic to $\mathbb{R}^2$ under the continuous bijective function $f: \mathbb{R} \rightarrow \mathbb{C}, f(x, y) = x + yi$, and is thus generally visualized as a coordinate plane with a real axis and an imaginary axis. From this homeomorphic relationship, the single-point compactification of $\mathbb{R}^2$ implies a single-point compactification of $\mathbb{C}$ with a point ∞. As above, this single-point compactification will be notated as $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.

Möbius Transformations

A Möbius transformation is a transformation which can be expressed as some combination of a translation, inversion, dilation, and rotation on the complex plane. In general, any Möbius transformation can be written as a function $f(z) = \frac{az+b}{cz+d}$ for some complex coefficients a, b, c, d, which satisfy $ad - bc \neq 0$. Some examples of Möbius transformation as given in *Möbius Transformations Revealed* are show in Figure 4 and Figure 5.

Figure 4

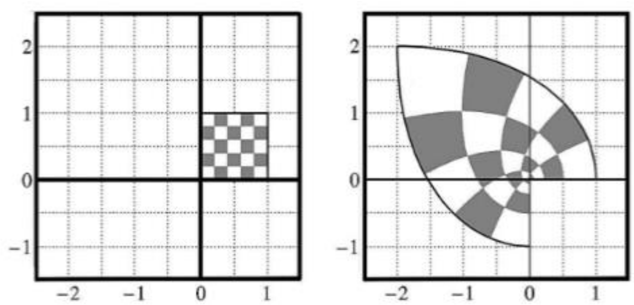
A Colored Rectangle and Its Image Mapped via $f(z) = cos(z)$



Note. A colored grid is shown before and after transformation. From "Möbius Transformations Revealed" article by D. N. Arnold and J. Rogness, 2008, *Notices of the AMS*. Copyright by the American Mathematical Society.

Figure 5

A Checkerboard on the Unit Square and Its Image Under $f(z) = z^3$



Note. A checkerboard-patterned grid shown before and after transformation. From "Möbius Transformations Revealed" article by D. N. Arnold and J. Rogness, 2008, *Notices of the AMS*. Copyright by the American Mathematical Society.

Additionally, Möbius transformations can be represented via stereographic projection. In particular, a Möbius transformation is the result of a projection of the complex plane onto the sphere, some movement of the sphere, and a projection from the sphere back onto the plane.

Previously, $s(X)$ was defined as the function mapping the sphere onto the plane through stereographic projection. Consider now the function $s^{-1}(X)$, which maps the plane onto the sphere. For simplicity, this inverse projection is defined with the sphere sitting atop the origin of the complex plane, with center at $(0, 0, 1)$.

As before, let N be the point on a sphere in the direction of $\vec{n} = \langle 0,0,1 \rangle$ from the center, i.e., $N = (0,0,2)$. Then for some complex number $x_0 + y_0 i$, represented by the coordinate $X = (x_0, y_0, 0)$, the image of $x + yi$ will be the intersection of the line NX with the sphere $x^2 + y^2 + (z - 1)^2 = 1$. The line NX is parameterized as:

$$x(t) = x_0 t$$

$$y(t) = y_0 t$$

$$z(t) = -2t + 2$$

Intersecting this line with the sphere $x^2 + y^2 + (z - 1)^2 = 1$ yields:

$$(x_0 t)^2 + (y_0 t)^2 + (-2t + 1)^2 = 1$$

$$x_0^2 t^2 + y_0^2 t^2 + 4t^2 - 4t + 1 = 1$$

$$t^2 (x_0^2 + y_0^2 + 4) = 4t$$

$$t = \frac{4}{x_0^2 + y_0^2 + 4}$$

Finally, substituting this value of t into the parameterized line NX gives the desired inverse projection.

$$s^{-1}(x,y,0) = \left(\frac{4x}{x^2+y^2+4}, \frac{4y}{x^2+y^2+4}, \frac{-8}{x^2+y^2+4}+2\right)$$

$$= \left(\frac{4x}{x^2+y^2+4}, \frac{4y}{x^2+y^2+4}, \frac{2x^2+2y^2}{x^2+y^2+4}\right).$$

This formulation will now aid in considering the result of various movements of the sphere between projections.

Lateral Translation

Suppose that after a projection of the plane onto the sphere, the sphere is translated so that it sits atop the point $(h,k,0)$ with center at $(h,k,1)$.

For any point $(x_0, y_0, 0)$, the image of this point after the projection onto this sphere is

$$s^{-1}(x_0,y_0,0) = \left(\frac{4x_0}{x_0^2+y_0^2+4}, \frac{4y_0}{x_0^2+y_0^2+4}, \frac{2x_0^2+2y_0^2}{x_0^2+y_0^2+4}\right).$$

As the sphere translates with vector $< h,k,0 >$, the image of this point becomes

$$s^{-1}(x_0,y_0,0)+(h,k,0) = \left(\frac{4x_0}{x_0^2+y_0^2+4}+h, \frac{4y_0}{x_0^2+y_0^2+4}+k, \frac{x_0^2+y_0^2}{2x_0^2+2y_0^2+4}\right).$$

Applying the stereographic projection back onto the plane, the final image of the point is

$$s\big(s^{-1}(x_0,y_0,0)+(h,k,0)\big) = s\left(\frac{4x_0}{x_0^2+y_0^2+4}+h, \frac{4y_0}{x_0^2+y_0^2+4}+k, \frac{2x_0^2+2y_0^2}{x_0^2+y_0^2+4}\right)$$

$$= \left(\frac{(1+1)\left(\frac{4x_0}{x_0^2+y_0^2+4}\right)}{1+1-\frac{2x_0^2+2y_0^2}{x_0^2+y_0^2+4}}+h, \frac{(1+1)\left(\frac{4y_0}{x_0^2+y_0^2+4}\right)}{1+1-\frac{2x_0^2+2y_0^2}{x_0^2+y_0^2+4}}+k, 0\right)$$

$$= \left(\frac{2(4x_0)}{2(x_0^2 + y_0^2 + 4) - (2x_0^2 + 2y_0^2)} + h, \frac{2(4y_0)}{2(x_0^2 + y_0^2 + 4) - (2x_0^2 + 2y_0^2)} + k, 0 \right)$$

$$= \left(\frac{8x_0}{8} + h, \frac{8y_0}{8} + k, 0 \right)$$

$$= (x_0 + h, y_0 + k, 0).$$

Thus, a lateral translation of the sphere between projections in the direction $\langle h, k, 0 \rangle$ exactly leads to a translation of the plane by the same vector $\langle h, k, 0 \rangle$. The corresponding function of this Möbius transformation takes the form $f(z) = z + b$ where $z = x + yi$ and $b = h + ki$, fitting the general form $f(z) = \frac{az+b}{cz+d}$ with $a = d = 1$ and $c = 0$.

Vertical Translation

Suppose that after a projection of the plane onto the sphere, the sphere is translated vertically with vector $\langle 0, 0, l \rangle$, placing the center at $(0, 0, l + 1)$.

For any point $(x_0, y_0, 0)$, the image after projection onto the sphere is and subsequent vertical translation is

$$s^{-1}(x_0, y_0, 0) + (0,0,l) = \left(\frac{4x_0}{x_0^2 + y_0^2 + 4}, \frac{4y_0}{x_0^2 + y_0^2 + 4}, \frac{2x_0^2 + 2y_0^2}{x_0^2 + y_0^2 + 4} + l \right)$$

Then projecting back onto the plane, the final image of the point is given by

$$s\big(s^{-1}(x_0, y_0, 0) + (0,0,l)\big) = s \left(\frac{4x_0}{x_0^2 + y_0^2 + 4}, \frac{4y_0}{x_0^2 + y_0^2 + 4}, \frac{x_0^2 + y_0^2}{x_0^2 + y_0^2 + 4} + l \right)$$

$$= \left(\frac{(l+2)\frac{4x_0}{x_0^2 + y_0^2 + 4}}{l + 2 - \left(\frac{2x_0^2 + 2y_0^2}{x_0^2 + y_0^2 + 4} + l \right)}, \frac{(l+2)\frac{4y_0}{x_0^2 + y_0^2 + 4}}{l + 2 - \left(\frac{2x_0^2 + 2y_0^2}{x_0^2 + y_0^2 + 4} + l \right)}, 0 \right)$$

$$= \left(\frac{(l+2)(4x_0)}{2(x_0^2 + y_0^2 + 4) - (2x_0^2 + 2y_0^2)}, \frac{(l+2)(4y_0)}{2(x_0^2 + y_0^2 + 4) - (2x_0^2 + 2y_0^2)}, 0 \right)$$

$$= \left(\frac{(l+2)(4x_0)}{8}, \frac{(l+2)(4y_0)}{8}, 0\right)$$

$$= \left(\frac{l+2}{2}x_0, \frac{l+2}{2}y + 0{,}0\right)$$

In terms of complex numbers, this means the image of $x + yi$ under this transformation is exactly $\frac{l+2}{2}(x+yi)$, representing a dilation of the plane with factor $\frac{l+2}{2}$. Then this is a Möbius transformation taking the form $f(z) = az$, fitting the general form $f(z) = \frac{az+b}{cz+d}$ with $b = c = 0$ and $d = 1$.

Lateral Rotation

Suppose that after a projection of the plane onto the sphere, the sphere is rotated about the z-axis by some angle θ. Note that this rotation about the z-axis will send a point (x_0, y_0, z_0) to $(x_0 \cos\theta - y_0 \sin\theta, x \sin\theta + y\cos\theta, z_0)$. Thus, after projection of a point $(x_0, y_0, 0)$ onto the sphere and subsequent rotation, the image of the point will be

$$\left(\frac{4x_0}{x_0^2+y_0^2+4}\cos\theta - \frac{4y_0}{x_0^2+y_0^2+4}\sin\theta, \frac{4x_0}{x_0^2+y_0^2+4}\sin\theta + \frac{4y_0}{x_0^2+y_0^2+4}\cos\theta, \frac{2x_0^2+2y_0^2}{x_0^2+y_0^2+4}\right).$$

After another stereographic projection onto the plane, this will become

$$\left(\frac{2\left(\frac{4x_0}{x_0^2+y_0^2+4}\cos\theta - \frac{4y_0}{x_0^2+y_0^2+4}\sin\theta\right)}{2-\frac{2x_0^2+2y_0^2}{x_0^2+y_0^2+4}}, \frac{2\left(\frac{4x_0}{x_0^2+y_0^2+4}\sin\theta + \frac{4y_0}{x_0^2+y_0^2+4}\cos\theta\right)}{2-\frac{2x_0^2+2y_0^2}{x_0^2+y_0^2+4}}, 0\right)$$

$$= \left(\frac{1}{8}(8x_0\cos\theta - 8y_0\sin\theta), \frac{1}{8}(8x_0\sin\theta + 8y_0\cos\theta), 0\right)$$

$$= (x_0\cos\theta - y_0\sin\theta, x_0\sin\theta + y_0\cos\theta, 0).$$

This is exactly the image of the point $(x_0, y_0, 0)$ after rotation about the z-axis with angle θ. Thus, the lateral rotation of the sphere corresponds exactly to a lateral rotation of the plane. In terms of the Möbius transformation on the plane, note that $(x + yi)(\cos\theta + i\sin\theta) = x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta)$, so in the general form $f(z) = \frac{az+b}{cz+d}$ this particular transformation occurs when $a = \cos\theta + i\sin\theta$, $b = c = 0$, and $d = 1$.

Longitudinal Rotation

Unlike lateral rotations, there are infinite possible axes through the center of the sphere upon which a longitudinal rotation could be performed. Note that the x, y, and z axes form an orthonormal basis of $\mathbb{R}^3$ as a list of orthonormal vectors with size equal to the dimension of $\mathbb{R}^3$ (Axler, 2017). Thus, it suffices to consider rotation about the lines parallel to the x and y $axes$.

Suppose that following a projection of the plane onto the sphere, the sphere is rotated by an angle θ about the line $y = 0, z = 1$. As before, the image of the point $(x_0, y_0, 0)$ projected onto the sphere is

$$\left(\frac{4x_0}{x_0^2 + y_0^2 + 4}, \frac{4y_0}{x_0^2 + y_0^2 + 4}, \frac{2x_0^2 + 2y_0^2}{x_0^2 + y_0^2 + 4}\right).$$

One may construct the rotation about $y = 0,\ z = 1$ by first translating the sphere with vector $\langle 0, 0, -1\rangle$, rotating on the x-axis, and translating again with vector $\langle 0, 0, 1\rangle$ before projecting the point back onto the plane.

After translating by $\langle 0, 0, -1\rangle$, the image of the point is

$$\left(\frac{4x_0}{x_0^2 + y_0^2 + 4}, \frac{4y_0}{x_0^2 + y_0^2 + 4}, \frac{x_0^2 + y_0^2 - 4}{x_0^2 + y_0^2 + 4}\right)$$

Rotating about the x-axis by angle θ:

$$\left(\frac{4x_0}{x_0^2+y_0^2+4}, \frac{4y_0\cos\theta-(x_0^2+y_0^2-4)\sin\theta}{x_0^2+y_0^2+4}, \frac{4y_0\sin\theta+(x_0^2+y_0^2-4)\cos\theta}{x_0^2+y_0^2+4}\right)$$

Translating by $\langle 0,0,1\rangle$:

$$\left(\frac{4x_0}{x_0^2+y_0^2+4}, \frac{4y_0\cos\theta-(x_0^2+y_0^2-4)\sin\theta}{x_0^2+y_0^2+4}, \frac{4y_0\sin\theta+(x_0^2+y_0^2-4)\cos\theta}{x_0^2+y_0^2+4}+1\right)$$

Finally, projecting the point back onto the plane and simplifying, the image of the point is given by:

$$\left(\frac{8x_0}{(x_0^2+y_0^2-4)(1-\cos\theta)-4y_0\sin\theta+8}, \frac{8y_0\cos\theta-2(x_0^2+y_0^2-4)\sin\theta}{(x_0^2+y_0^2-4)(1-\cos\theta)-4y_0\sin\theta+8}, 0\right)$$

Note that when $\theta = 180°$, this formulation simplifies exactly to $\left(\frac{4x_0}{x_0^2+y_0^2}, \frac{-4y_0}{x_0^2+y_0^2}\right)$. Comparing the original point to this image, the rotation has effectively dilated the plane by a factor of 4, performed an inversion about the unit circle, and reflected across the x-axis.

As a Möbius function, this can be represented by $f(z) = \frac{4}{z}$.

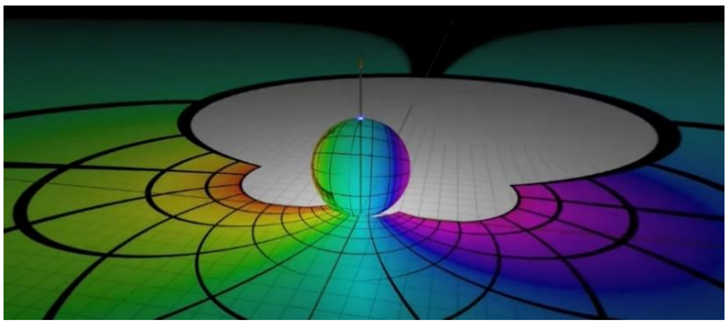
Notably this is the only movement of the sphere that results in the image of the point ∞ to be somewhere other than ∞. For the 180° rotation, the image of ∞ becomes exactly the point 0. For a partial inversion, represented by some angle $\theta \neq 180°$, the infinity point approaches 0 as θ approaches 180°.

Geometric Examination of Inversion

Although the algebraic relationship found above gives the image of the point ∞ and corresponding inversion about the origin under the special case of a 180° rotation, it is very difficult to work with for other angles of rotation. One might conjecture that the pattern will hold for other angles: an inversion about some point and circle followed by a reflection across some axis. Figure 6 shows the result of one such rotation, appearing to support this conjecture.

Figure 6

Longitudinal Rotation

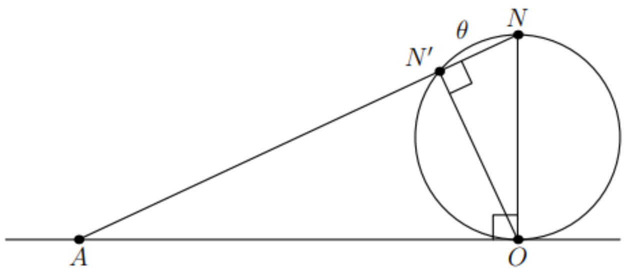


Note. The sphere from Figure 1 has been rotated longitudinally, showing the resulting projection on the plane below. From "Möbius Transformations Revealed," video by D. N. Arnold and J. Rogness, 2007. Copyright 2007 by the University of Minnesota.

Consider the cross-section formed by intersecting the plane $x = 0$ with the sphere and the complex plane, shown in Figure 7. Let N' be the image of N on the sphere after rotation by angle θ. Let A be the image of N' after stereographic projection.

Figure 7

Cross-Section of the Sphere After Rotation

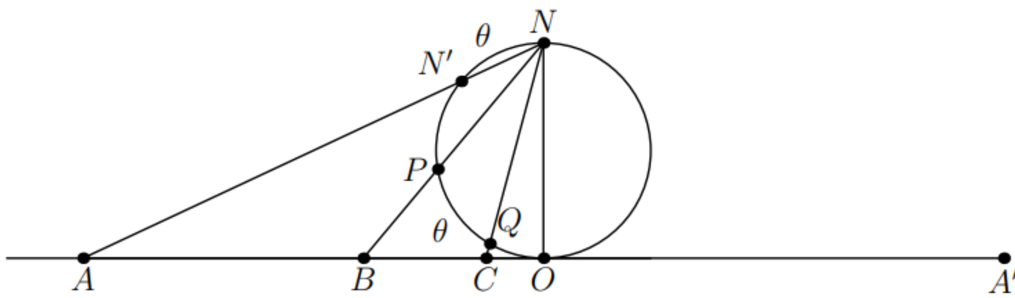


By Inscribed Angle Theorem, $\angle NON' = \frac{\theta}{2}$. Further, $\angle NAO = 90 - \angle ANO = \angle NON'$ so $\angle NAO = \frac{\theta}{2}$ as well. Since $NO = 2$ and $\angle NOA = 90°$, it follows that $AO = 2\cot\frac{\theta}{2}$.

Note that as $s(N) = \infty$, A is exactly the image of ∞ on $\hat{\mathbb{C}}$ after the projections and rotation of the sphere. It has been conjectured previously that this transformation will result in some group of inversion and reflection. If this conjecture is true, then A is a point that has been inverted from ∞ and then reflected across the origin O. Let A' be the reflection of A across O, as shown in Figure 8. Let P be a point on the sphere with image Q after rotation. Let B and C be points on $\hat{\mathbb{C}}$ such that $B = s(P)$ and $C = s(Q)$.

Figure 8

Additional Points on the Rotating Sphere



Inscribed Angle Theorem gives $\angle PNQ = \frac{\theta}{2}$. Let $\angle BNO = \phi$, implying $\angle CNO = \phi - \frac{\theta}{2}$. Then right triangles $\triangle BNO$ and $\triangle CNO$ allow the expressions of BO and CO as:

$$BO = 2\tan\phi$$

$$CO = 2\tan\left(\phi - \frac{\theta}{2}\right)$$

If the conjecture is correct, then A' is the image of ∞ after inversion and therefore the center of inversion. Let $C\,'$ be the reflection of C across O, then the conjecture will be proven true if $C\,'$ is the image of B about a circle centered at A' with radius dependant only on θ.

$$A'B \cdot A'C' = (A'0 + BO)(A'O - C'O)$$

$$= (AO + BO)(AO - CO)$$

$$= \left(2\cot\frac{\theta}{2} + 2\tan\phi\right)\left(2\cot\frac{\theta}{2} - 2\tan\left(\phi - \frac{\theta}{2}\right)\right)$$

$$= 4\left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} + \frac{\sin\phi}{\cos\phi}\right)\left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{\sin\left(\phi - \frac{\theta}{2}\right)}{\cos\left(\phi - \frac{\theta}{2}\right)}\right)$$

$$= 4\left(\frac{\cos\frac{\theta}{2}\cos\phi + \sin\frac{\theta}{2}\sin\phi}{\sin\frac{\theta}{2}\cos\phi}\right)\left(\frac{\cos\frac{\theta}{2}\cos\left(\phi - \frac{\theta}{2}\right) - \sin\frac{\theta}{2}\sin\left(\phi - \frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\cos\left(\phi - \frac{\theta}{2}\right)}\right)$$

$$= 4\left(\frac{\cos\left(\frac{\theta}{2} - \phi\right)}{\sin\frac{\theta}{2}\cos\phi}\right)\left(\frac{\cos\phi}{\sin\frac{\theta}{2}\cos\left(\phi - \frac{\theta}{2}\right)}\right)$$

$$= \frac{4}{\sin^2\frac{\theta}{2}}$$

$$= 4\csc^2\frac{\theta}{2}$$

This product is independent of ϕ, thus the result of a θ-angled longitudinal rotation can be characterized in the following way: first, the plane is inverted about a circle with radius $2\csc\frac{\theta}{2}$ center A', where A' lies on the longitudinal plane of inversion, with a distance of $2\cot\frac{\theta}{2}$ from the origin. Then, the plane is reflected across the line passing through the origin perpendicular to vector $\overrightarrow{OA'}$.

CHAPTER III

THE SPLIT-COMPLEX PLANE

Introduction

The split-complex plane $\mathbb{R}^{1,1}$ is defined by the split-complex numbers, which follow the form:

$$x + yj \quad | \, x, y \in \mathbb{R}, \, j^2 = 1, j \neq 1, -1.$$

As with the complex plane, the split-complex plane is homeomorphic to $\mathbb{R}^2$ and so the same single-point compactification $\widehat{\mathbb{R}}^{1,1} = \mathbb{R}^{1,1} \cup \infty$ exists. On the split-complex plane, magnitude of a split-complex number is defined by Emanuello and Nolder (2014) as

$$|x + yj|^2 = |x^2 - y^2|.$$

This differs from the usual notion of magnitude as equivalent to a straight-line distance but maintains desired properties of magnitude toward split-complex numbers. It can be verified, for example, that under this definition of magnitude $|zw| = |z| \cdot |w|$ for split-complex numbers z and w. Let $z = a + bj$ and $w = c + dj$. Then according to the definition $|x + yj| = |x^2 - y^2|$,

$$|zw|^2 = |(a + bj)(c + dj)|^2$$

$$= |ac + bcj + adj + bdj^2|^2$$

$$= |(ac + bd) + (bc + ad)j|^2$$

$$= |(ac + bd)^2 - (bc + ad)|^2$$

$$= |a^2c^2 + b^2d^2 + 2abcd - (b^2c^2 + a^2d^2 + 2abcd)|$$

$$= |a^2c^2 + b^2d^2 - b^2c^2 - a^2d^2|$$

$$= |(a^2 - b^2)(c^2 - d^2)|$$

$$= |z|^2|w|^2$$

$$|zw| = |z||w|.$$

So, the desired property of magnitude holds. Note that the nature of dilations is preserved under this defined magnitude. Let $z = x + yj$ and $m \in \mathbb{R}$ such that $m > 0$. Then $mz = (mx) + (my)j$ and $|mz|^2 = |(mx)^2 - (my)^2| = |m^2(x^2 - y^2)| = m^2|z|^2$. It follows that $|mz| = m|z|$, thus dilating the plane with factor m is equivalent to multiplying each component of the split-complex numbers z by m.

The above properties will educate the nature of inversion on $\mathbb{R}^{1,1}$. Inversion about a circle with radius r centered at the origin takes a number z to a number $z\,'$ on the same ray from the origin such that $|z||z'| = r^2$. Since z and $z\,'$ are on the same ray from 0, there exists some real $m > 0$ such that $z\,' = mz$. Let $z = x + yj$ and $z\,' = m(x + yj) = (mx) + (my)j$.

$$r^2 = |z||z'|$$

$$= |z||mz|$$

$$= |mz^2|$$

$$= m|z|^2$$

$$= m|x^2 - y^2|$$

$$m = \frac{r^2}{x^2 - y^2}$$

$$= \frac{r^2}{|z|^2}.$$

Hence, the image of z under this inversion is $\frac{r^2}{|z|^2}z$, as it is in complex numbers.

Translation and rotation will clearly act the same on $\mathbb{R}^{1,1}$ as they do on $\mathbb{C}$, so the result is that all Möbius transformations will act on the split-complex plane identically as they do on the complex plane. This further implies that each of these transformations can be expressed geometrically by the same stereographic projections as in the previous chapter. However, it should be noted that the same general function of Möbius transformations will not translate to the split-complex plane. Rotation by an angle θ about the origin is easily expressed in complex numbers by multiplication by $cos\,\theta + i\,sin\,\theta$. This is not the case for split-complex numbers.

In general, the form $f(z) = az + b$ may be used to characterize Möbius functions which are a combination of dilation and translation, but rotation and inversion do not have such a straightforward representation at this time.

Laguerre Transformations

As not all Möbius transformations translate well to the split-complex plane, it warrants exploration what sorts of transformations occur under the same algebraic form $f(z) = \frac{az+b}{cz+d}$. In general, this is called a Laguerre transformation. Let a, b, c, d be split-complex numbers on $\mathbb{R}^{1,1}$ with $ad - bc \neq 0$.

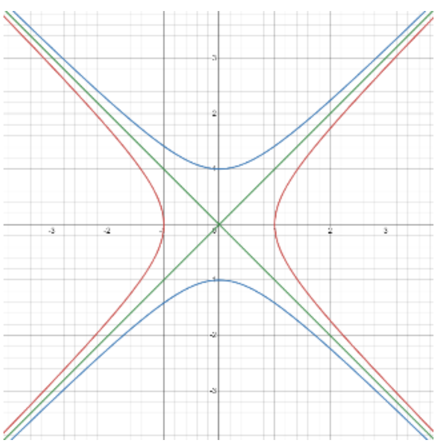
Laguerre Transformations Represented by Multiplication

If a is a real number, then as noted previously $f(z) = az$ results in a dilation of the plane centered at 0. In order to explore multiplication of split-complex numbers when $a, z \notin \mathbb{R}$ it is necessary to have some other representation of the split-complex numbers. Within complex numbers, numbers with the same magnitude lie on a circle and as a result the complex numbers lend well to the polar form $z = |z|(\cos\theta + i\sin\theta)$. Within the split-complex numbers, let $c \in \mathbb{R}$ be a constant and consider all complex numbers $z = x + yj$ such that $|z|^2 = c$. Then the components x and y are characterized by $x^2 - y^2 = c$, thus the set of such split-complex numbers takes the form of a hyperbola. As a result, the analog of the polar form among split-complex numbers should be the hyperbolic polar form.

It should be noted that the hyperbolic polar form has three different representations depending on where a split-complex number lies. Figure 9 shows two unit hyperbolas and their asymptotes.

Figure 9

Hyperbolic Polar Form



Along the red hyperbola are all split-complex numbers $z = x + yj$ where $x^2 - y^2 = 1$, let H_1 be the set of numbers obtained by scaling this hyperbola by any real number r. These numbers are represented by $z = r(cosh\ \phi + jsinh\ \phi)$. Along the blue hyperbola are all split-complex numbers where $x^2 - y^2 = -1$, likewise let H_2 be the scales of these numbers. They are represented by $z = r(\sinh \phi + j \cosh \phi)$. Along the green asymptotes are the split-complex numbers where $x^2 - y^2 = 0$, let this be A. These numbers are represented by $z = r(1 \pm j)$. Each of these must be considered separately for the purpose of understanding multiplication and division of split-complex numbers.

Case 1: $a, z \in H_1$

Suppose both a and z lie on H_1. Let $z = r(\cosh \phi + j \sinh \phi)$ and $a = s(\cosh \theta + j \sinh \theta)$. The result of multiplying these numbers is shown below.

$$az = r(\cosh \phi + j \sinh \phi) \cdot s(\cosh \theta + j \sinh \theta)$$

$$= rs\big((\cosh \phi \cosh \theta + \sinh \phi \sinh \theta) + j(\cosh \phi \sinh \theta + \sinh \phi \cosh \theta)\big)$$

$$= rs(\cosh(\phi + \theta) + j \sinh(\phi + \theta))$$

This is exactly z scaled by a factor of s and given a hyperbolic rotation by θ, with the product lying on H_1.

Case 2: $a, z \in H_2$

Let $z = r(\sinh \phi + j \cosh \phi)$ and $a = s(\sinh \theta + j \cosh \theta)$. Then multiplication yields:

$$az = r(\sinh \phi + j \cosh \phi) \cdot s(\sinh \theta + \cosh \theta)$$

$$= rs\big((\sinh \phi \sinh \theta + \cosh \phi \cosh \theta) + j(\cosh \phi \sinh \theta + \sinh \phi \cosh \theta)\big)$$

$$= rs\,(\cosh(\phi + \theta) + j(\sinh(\phi + \theta))$$

This again represents z scaled by a factor of s and given a hyperbolic rotation by θ. Note that the resulting product lies on H_1 despite both a and z lying on H_2.

Case 3: $z \in H_1$, $a \in H_2$

Let $z = r(\cosh \phi + j \sinh \phi)$ and $a = s(\sinh \theta + j \cosh \theta)$. Then multiplication yields:

$$az = r(\cosh \phi + j \sinh \phi) \cdot s(\sinh \theta + j \cosh \theta)$$

$$= rs(\cosh \phi \sinh \theta + \sinh \phi \cosh \theta) + j(\sinh \phi \sinh \theta + \cosh \phi \cosh \theta))$$

$$= rs(\sinh(\phi + \theta) + j\cosh(\phi + \theta))$$

As expected, the result is that z is scaled by factor s and given a hyperbolic rotation by θ. The resulting product lies on H_2. Reversing a and z shows an identical result for $z \in H_2$, $a \in H_1$.

Case 4: $a \in A$

Suppose a lies on one of the asymptotes, i.e., $a = s(1 \pm j)$. Then for any $z = x + yj$,

$$az = (x + yj)(1 \pm j)$$

$$= (x \pm y) + j(\pm x + y)$$

$$= (x \pm y) \pm j(x \pm y)$$

Thus multiplying by any $a \in A$ results in a product in A.

In summary, for any split-complex numbers not contained in the asymptotes $y = \pm x$, multiplication results in a hyperbolic rotation along with a scaling consistent with $|zw| = |z||w|$. Additionally, multiplying by a split-complex number in H_1 will not change the region of the multiplicand while multiplying by a split-complex number in H_2 will change the region. Multiplying by a number in A collapses the plane into A.

Laguerre Transformations Represented by Division

It has been shown above that multiplication results in a hyperbolic rotation so for $c \in \mathbb{R}^{1,1}$, cz represents a hyperbolic rotation of z. Addition has been previously stated to result in a translation, so for $d \in \mathbb{R}^{1,1}$ the divisor $cz + d$ is a transformed z via hyperbolic rotation and translation. Let $w = cz + d$ be represented by $w = r(\cosh\phi + j\sinh\phi)$, then the general Laguerre transformation can be stated as:

$$f(z) = \frac{az + b}{cz + d}$$

$$= \frac{az + b}{w}$$

$$= \frac{az + b}{r(\cosh \phi + j \sinh \phi)}$$

$$= \frac{(az + b)(\cosh \phi - j \sinh \phi)}{r}$$

$$= \frac{(az + b)(\cosh(-\phi) + j \sinh(-\phi))}{r}$$

This division then represents taking the complex number $az + b$ and applying a hyperbolic rotation by $-\phi$ and scale by $\frac{1}{r}$. $az + b$ is itself the result of a hyperbolic rotation by the hyperbolic angle of a and translation by b. This gives some characterization for the Laguerre transformations as a combination of hyperbolic rotations, scaling by real number factor, and translation by some vector.

It is much more involved, and remains to be shown, what exact sort or rotation and scaling $w = cz + d$ gives to the numerator $az + b$ as the hyperbolic angle and magnitude of w depend on z. The final result is that the differing hyperbolic rotation and scaling dependent on z results in an axial inversion (Yaglom, 1968) similar to the resulting inversion for Möbius functions in the complex plane.

CHAPTER IV

THE HYPERBOLIC PLANE

Introduction

Hyperbolic geometry stems from dissatisfaction with Euclid's fifth axiom, commonly known as the *parallel axiom*. While the first four of Euclid's axioms are simple statements in simple terms, such as "a line can be extended indefinitely," the fifth postulate is stated by Euclid in a much more complex manner:

> [Let it be assumed] that if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Stillwell, 1989)

There are many results of this assumption that might simplify the statement, most notable is the converse of the statement. If the first straight line meets the other two with two angles on one side, which add to exactly 180 degrees (i.e., two right angles) then the lines will not intersect and are instead parallel. Further, one could state that given one line L and a point P, which is not on L, the axiom implies the existence of a unique line through P which is parallel to L.

The basis for hyperbolic geometry comes from the following negation of Euclid's fifth axiom: P_2 - given a line L and point P not on L, there are at least two lines through P, which do not meet L.

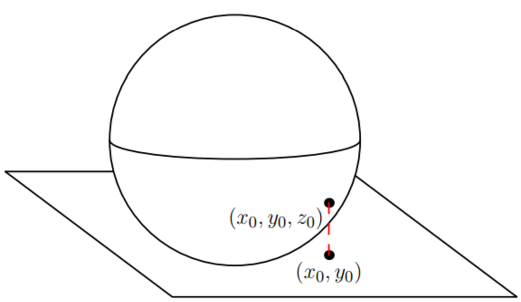
The effect that this negation has on lines is that by nature for two distinct lines to pass through a point and both not intersect a third line, the lines themselves must curve away from each other. Any plane with geometry that follows the axiom P_2 is said to be a hyperbolic plane. Many constructions of hyperbolic planes exist, with varying properties. In the next section, one such hyperbolic plane will be constructed and hereafter referred to as $\mathbb{H}^2$. The idea in the construction is to take a transformation on $\mathbb{R}^2$ using the sphere sitting atop the origin in such a way that the images of lines in $\mathbb{R}^2$ will become lines in $\mathbb{H}^2$, which curve away from one another.

Constructing a Model Through Central Projection

For some subset $S \subset \mathbb{R}^2$, let $r > 0$ be large enough that the disk of radius r centered at the origin covers S. A sphere of radius r is placed atop the plane, tangent to the plane at the origin. For any point $A \in \mathbb{R}^2$, the point is first projected upward onto the lower hemisphere, let the image of (x_0, y_0) under this first projection be (x_1, y_1), as shown in Figure 10.

Figure 10

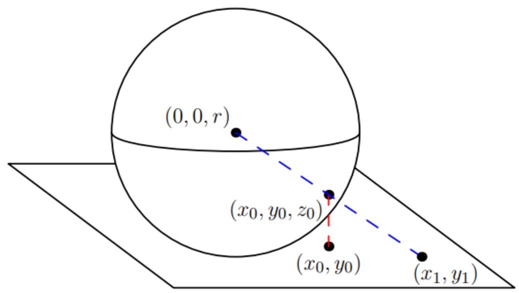
Lateral Projection Onto the Sphere



Next a projection is performed from the center of the sphere, mapping (x_1, y_1) to a point (x_2, y_2) on the plane below, as shown in Figure 11.

Figure 11

Central Projection Onto the Plane



Algebraic Expression of the Model

The model consists of two projections, which are defined algebraically as follows. First, to place a point (x_0, y_0)) from the plane onto the lower hemisphere we simply intersect the line $x = x_0, y = y_0$ with the sphere $x^2 + y^2 + (z - r)^2 = r^2$. Thus define the function $f: \mathbb{R}^2 \to \mathbb{R}^3$ by $f(x, y) = \left(x, y, r - \sqrt{r^2 - x^2 - y^2}\right)$.

The next projection extends a line from the center of the sphere $(0,0,r)$ through the point (x_0, y_0, z_0) and to intersect with the plane $z = 0$. This line can be parameterized as $x = x_0 t,\ y = y_0 t,\ z = (z_0 - r)t + r$. Plugging $z = 0$ into the previous equation, yields $t = \frac{r}{r - z_0}$. Plugging this into the other two equations gives the function $g: \mathbb{R}^3 \to \mathbb{R}^2$ as $g(x, y, z) = \left(\frac{xr}{r-z}, \frac{yr}{r-z}\right)$.

Finally, combining these two functions gives the transformation.

$$(g \circ f)(x,y) = \left(\frac{x}{\sqrt{r^2 - x^2 - y^2}}, \frac{y}{\sqrt{r^2 - x^2 - y^2}}\right).$$

Hereafter, the combined transformation will be referred to as $h = g \circ f$

Note that this transformation scales a point (x, y) from the origin. However, this does not represent a dilation of the plane as the scale factor is dependent on the distance from (x, y) to the origin, i.e. $|(x,y)| = \sqrt{x^2 + y^2}$. Consider the vector $\vec{V}$, the transformation gives

$$h(\vec{V}) = \frac{1}{\sqrt{r^2 - |V|^2}}\vec{V}$$

Consider the effect this transformation has on a line. Since the vector scaling is not a uniform dilation, the shape of the line must be warped in some way. However, since the scale factor of any particular vector under this transformation depends only on the distance from the point to the origin, the orientation of a line can be disregarded as one explores the effect of the transformation under the line. Let the shortest distance between a line l and the origin be c. Then there exists a rotation of l about the origin, which places l onto the line $x = c$. Thus, it suffices to consider the result of the transformation h on $x = c$. Let (c, t) be a point on $x = c$, with c being a constant and t a parameter. Then $h(c,t) = \left(\frac{c}{\sqrt{r^2-c^2-t^2}}, \frac{t}{\sqrt{r^2-c^2-t^2}}\right)$. Then the resulting functions $x = x(t)$ and $y = y(t)$ give the following result.

$$y = \frac{t}{\sqrt{r^2 - c^2 - t^2}}$$

$$y^2 = \frac{t^2}{r^2 - c^2 - t^2}$$

$$(r^2 - c^2)y^2 - t^2y^2 = t^2$$

$$(r^2 - c^2)y^2 = (y^2 + 1)t^2$$

$$t^2 = \frac{(r^2 - c^2)y^2}{y^2 + 1}$$

$$x = \frac{c}{\sqrt{r^2 - c^2 - t^2}}$$

$$x^2 = \frac{c^2}{r^2 - c^2 - t^2}$$

$$x^2 = \frac{c^2}{r^2 - c^2 - \frac{(r^2 - c^2)y^2}{y^2 + 1}}$$

$$= \frac{c^2(y^2 + 1)}{r^2 - c^2}$$

$$x^2 - \frac{y^2}{r^2 - c^2} = \frac{c^2}{r^2 - c^2}$$

$$\frac{x^2}{c^2/(r^2 - c^2)} - y^2 = 1.$$

This implies the image of the line $x = c$ is exactly a hyperbola. Notably the eccentricity of the hyperbola is given by $\sqrt{1 + \frac{1}{c^2/(r^2 - c^2)}} = \frac{r}{c}$. Thus for a fixed radius r, the eccentricity of the image of a line is inversely proportional to the distance from the line to the origin.

Geometric Interpretations

Some inferences can be made based on the algebraic relationship found above. First, suppose $D\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a dilation of the plane centered at the origin with some factor d.

Then any vector $\vec{V}$ is scaled by the dilation $d\vec{V}$ and the result of our hyperbolic transformation on the scaled vector would yield $h(d\vec{V}) = \frac{d}{\sqrt{r^2 - d|\vec{V}|^2}}\vec{V} \neq d \cdot h(\vec{V})$.

Thus, a dilation on $\mathbb{R}^2$ followed by the hyperbolic transformation does not lead to a dilation of the hyperbolic plane. Instead, consider the result of our exploration on eccentricity. The dilation D scales the distance from the origin to any line. Let l be a line with shortest distance from the origin equal to d, then $D(l)$ has a distance to the origin of dc. The result after our hyperbolic transformation then is a hyperbola with eccentricity $\frac{r}{dc} = \frac{r}{c} \div d$. It follows that a dilation on $\mathbb{R}^2$ prior to the hyperbolic transformation is equivalent to a uniform change in eccentricity of the hyperbolic curves on $\mathbb{H}^2$. A dilation on $\mathbb{H}^2$ can instead be performed by a movement of the sphere between projections, as it was on the complex plane.

It was previously noted the transformation on individual points is a linear transformation from the origin, thus any rotation about the origin prior to the hyperbolic transformation will result in a rotation of $\mathbb{H}^2$. This can also be achieved by rotating the sphere between projections.

CHAPTER V

CONCLUSION

Conclusion

Stereographic projection has been used here to demonstrate the compactification of several two-dimensional planes. The complex plane was explored in detail with the Möbius transformations and their algebraic representation. The nature of different movements on the sphere between stereographic projections has been explored and relationships found between those movements and transformations on the plane. In all cases, a lateral translation or rotation of the sphere resulted in exactly the same transformation on the plane and a vertical translation resulted in a dilation with the factor linearly dependent on the magnitude of the translation.

Of particular note is the discovered relationship of longitudinal rotation of the sphere with a particular inversion and reflection of the plane. The center and radius of inversion has been found in terms of the angle of rotation on the sphere.

Transformations on the split-complex plane resulting from movements on the sphere have been noted to be identical to those on the complex plane. However, the general form of the Möbius functions fails to translate to the split-complex plane as split-complex numbers do not have the same rotational properties of complex numbers.

A hyperbolic plane has been constructed using a central projection instead of a stereographic projection. The projection maps the lines in $\mathbb{R}^2$ onto parabolas in $\mathbb{H}^2$. Lateral rotations and translations on the sphere represent identical rotations and translations on the plane. A vertical translation of the sphere gives a conventional dilation of $\mathbb{H}^2$. This differs from the result of dilating $\mathbb{R}^2$ prior to projection, which instead results in a uniform change in eccentricity of the hyperbolas in $\mathbb{H}^2$.

Limitations of the Study

This study focused mostly on transformations and relationships that could be obtained by movement of a sphere before and after projection. As such, certain algebraic forms of Möbius transformations have not been fully explored, in particular the full implications of what sort of movements of the sphere result from different values of c and d in the general form $f(z) = \frac{az+b}{cz+d}$. Laguerre transformations on the split-complex plane were discussed in slightly more detail algebraically but ultimately the same denominator involving c and d requires further exploration in that plane. A generalization was given by Yaglom (1968) but not fully proved here.

The hyperbolic plane warrants further investigation, especially with respect to longitudinal rotations and their effect on the plane. It is suspected that this would result in another sort of inversion on the plane but proof requires further study.

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