TEXAS WOMAN'S UNIVERSITY

Mathematics and Computer Science

Investigations of the Fitzhugh-Nagumo Neuron Model

Christian Olmeda, Michelle Lobb, Mosammat Yeasmin, Faculty Sponsor: Dr. Ellina Grigorieva

Abstract

Neurons are cells in the body that transmit information to the brain and the body by amplifying an incoming stimulus (electrical charge input) and tranmitting it to neighboring neurons, then turning off to be ready for the next stimulus. These cells also have fast and slow mechanisms to open ion channels in response to electrical charges.

Neurons use changes of sodium and potassium ions across the cell membrane to amplify and transmit information. *Voltage-gated channels* exist for each kind of ion, which open and close in response to voltage difference, and are closed in a resting neuron. If the electrical excitation reaches a sufficiently high level, called an action potential, the neuron fires and transmits the excitations to other neurons.

In this work we will model neuron action by the following nonlinear system of differential equations:

$$\begin{cases} \dfrac{dv}{dt} = -v(v-a)(v-1) - w \\ \dfrac{dw}{dt} = \varepsilon(v - \xi w) \end{cases}$$

Eq (1)

Here **v(t)** is the potential. We let $v=a$ be the potential above which the neuron fires and $v=1$ the potential at which sodium channel opens ($0<a<1$) ; **w(t)** denotes the strength of the blocking mechanism.

Finding Equilibrium Points

$v=x$, $w=y$, $\xi=c$ and $\varepsilon=b$

$-x(x-a)(x-1)-y=0$

$b(x-cy)=0 \Rightarrow y=x/c$

$x(cx^2-(1+a)cx+ac+1)=0$

$x=0$ $y=0$ is an equilibrium point

To ensure that other equilibrium points exist, the discriminant of the remaining quadratic function must be >0 or =0

$D= (1+a)^2c^2-4(ac+1)c= (1+2a+a^2)c^2-4ac^2-4c$

$D=(c-ac)^2-4c \Rightarrow (c-ac)^2-4c = 0$

$c=4/(1-a)^2$

If $c > 4/(1-a)^2$ then two real roots exist

If $c < 4/(1-a)^2$ then two complex roots exist

If $c=4/(1-a)^2$ then only one root exists, and is repeating

What happens if a=5, b=4, and c=3?

Using the quadratic equation and y=x/c to find the other two equilibrium points.

$X_1=1.1$, $Y_1=0.35$

$X_2=4.9$, $Y_2=1.6$

Mathematical Behavior

$$J_{(x,y)} = \begin{bmatrix} -3x^2+2(1+a)x-a & -1 \\ b & -bc \end{bmatrix} \qquad J_{(0,0)} = \begin{bmatrix} -a & -1 \\ b & -bc \end{bmatrix}$$

At this point in the investigation, we used Jacobian matrices to linearize the system of differential equations so that the behavior of the equilibrium can be investigated.

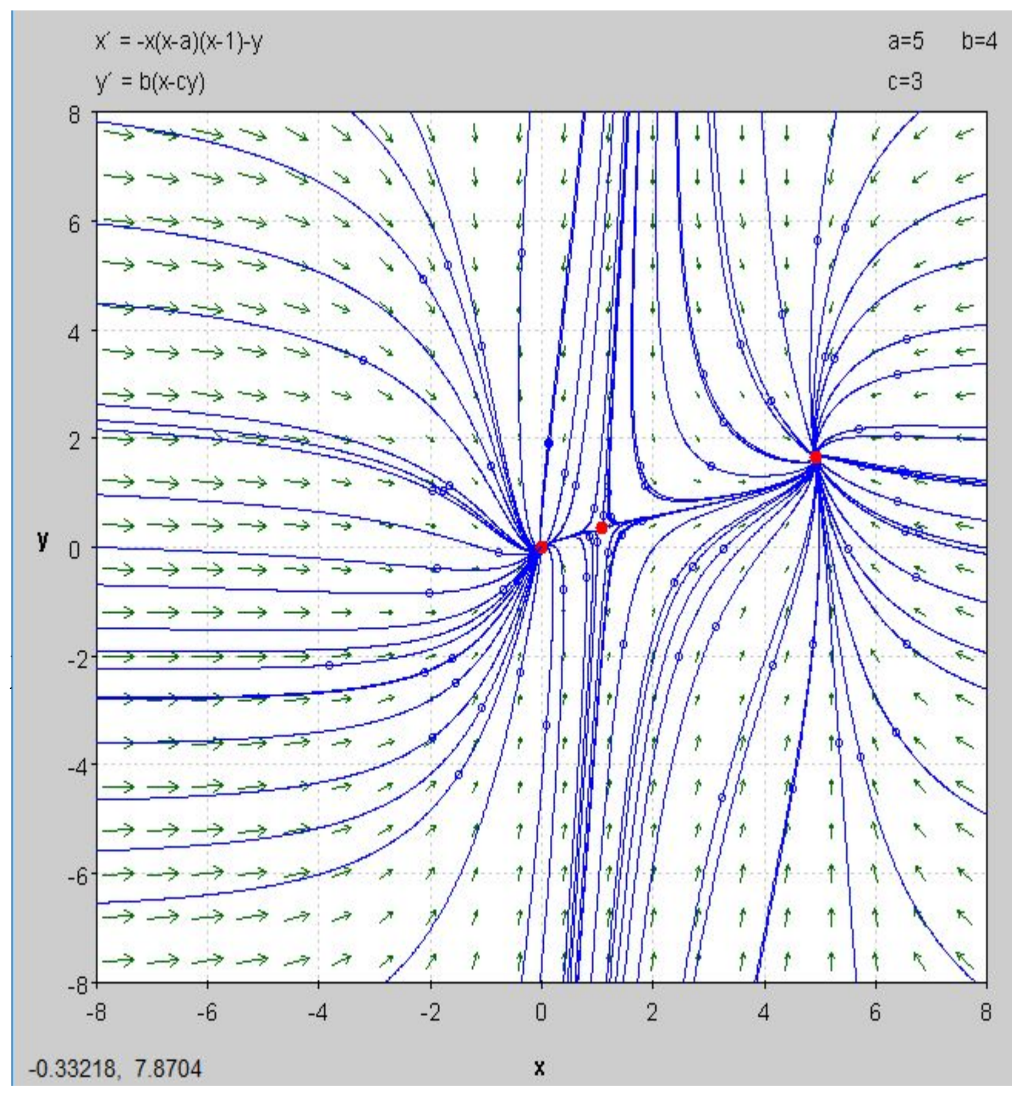
$$\lambda_{1,2} = \frac{-a - bc \pm \sqrt{(a+bc)^2 - 4(abc+b)}}{2}$$

For point (0,0), both λ_1 and λ_2 are less than 0, indicating a sink, and is asymptotically stable.

For point (1.1, 0.35) the λ values have opposite signs, indicating a saddle point.

For point (4.9, 1.6) both λ_1 and λ_2 are less than 0, indicating a sink, and is asymptotically stable.

These are all observed in the phase plane of this system, as shown below. The red indicators represent the aforementioned equilibrium points.



It can be observed that depending on the initial condition the behavior of the curve will change. By investigating the found equilibrium points it can be seen that (0,0) and (4.9,1.6) are functioning as sinks. Curves are flowing into them. (1.1, 0.35) is functioning as a saddle point. It is redirecting curves away from itself.

Behavior in Reality

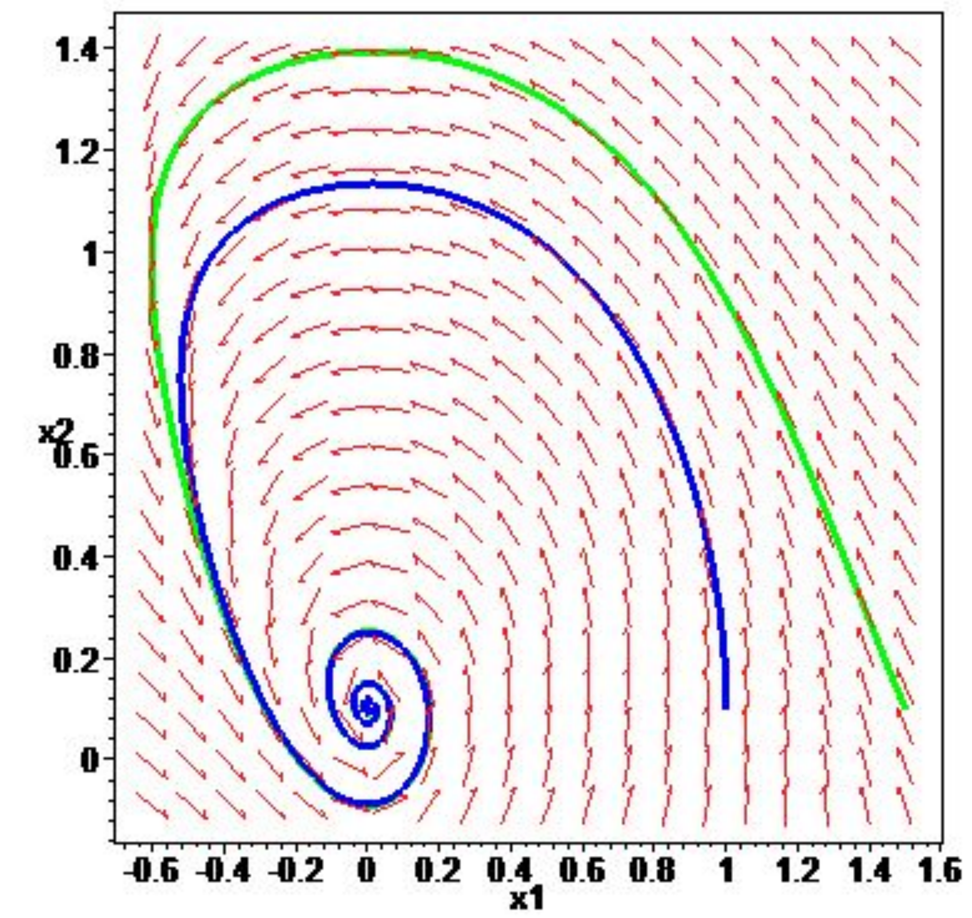
The constraints of reality drastically change the behavior of the system. Reality dictates that our values do not allow for the existence of 3 equilibrium points but of only 1 equilibrium point. The equation that was utilized to find the 2 additional equilibrium points

$c<4/(1-a)^2$

which indicates that the two additional roots found in the original approach to the model do no exist in reality.

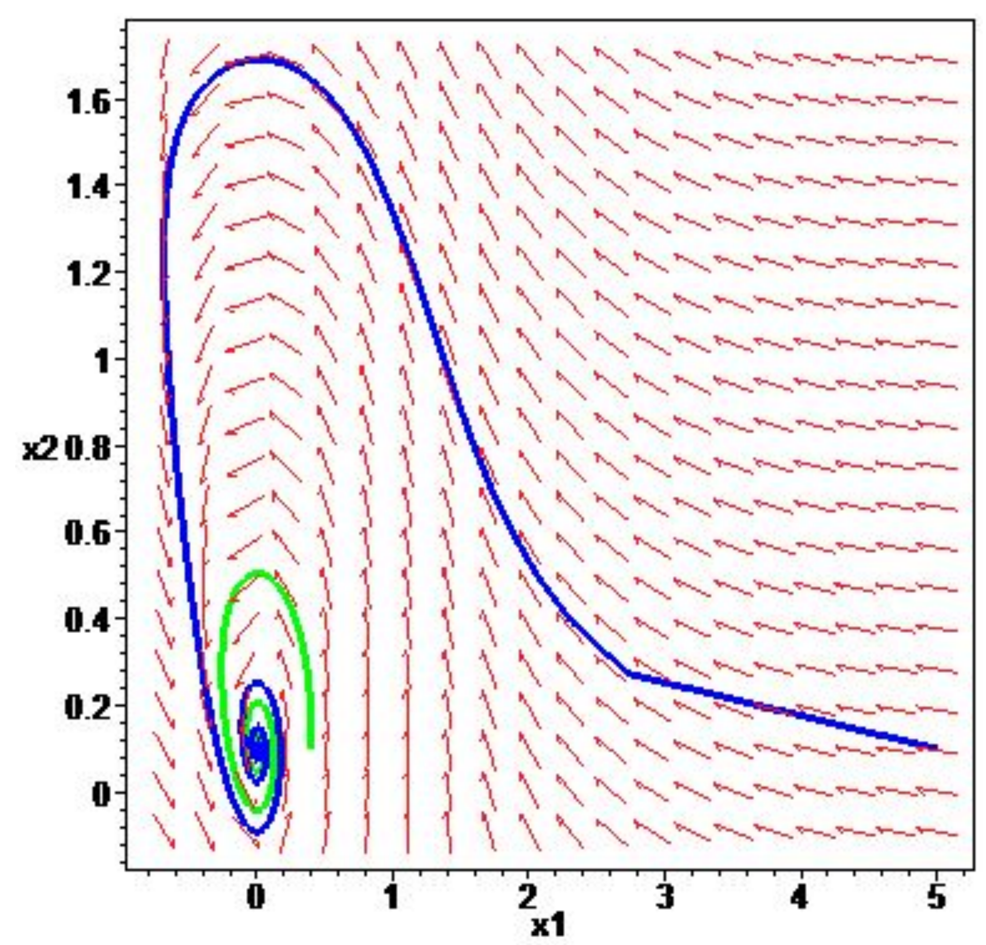
Conditions for blue curve:
a=0.3, b=1, c=0.01, and $v_0=1$

Conditions for green curve:
a=0.3, b=1, c=0.01, and $v_0=1.5$



Using the same initial potential two radically different curves are produced. These two cures follow the same behavior as each other as opposed to their counterparts in the other model. This holds true for all initial potentials.

Same initial conditions as before except $v_0= .4$ and $v_0=5$



Addition of a constant electrical current

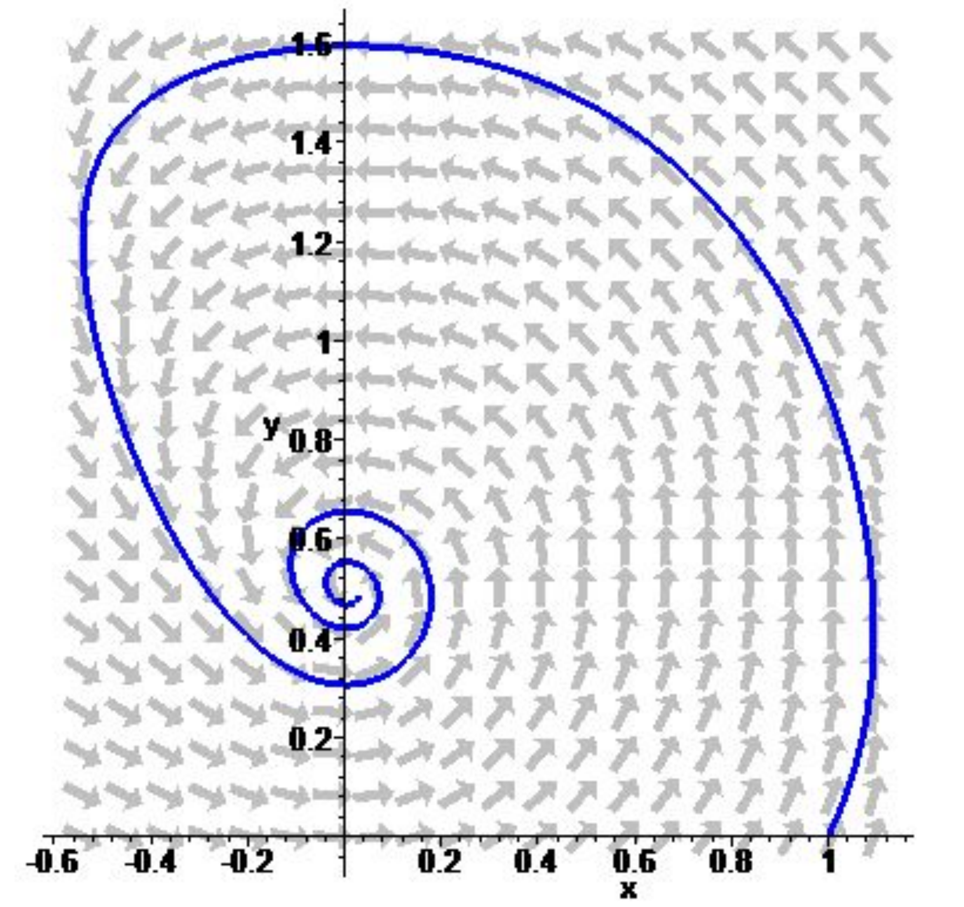
The model can be adjusted to account for a constant input of positive ions instead of a single pulse. This alters the location of the single equilibrium point, pushing it into the first quadrant.

When the system is affected by a constant electrical current, hereby referred to as J, as opposed to the singular pulse, the behavior of the system changes dramatically. In order to model this mathematically, we add J to the rate of change of potential resulting in the following nonlinear system:

$$\begin{cases} \dfrac{dv}{dt} = -v(v-a)(v-1) - w + J \\ \dfrac{dw}{dt} = \varepsilon(v - \xi w) \end{cases}$$

Eq(2)

Conditions for curve:
a=0.3, b=1, c=0.01, J=0.5 and $v_0=1$



Conditions for curve:
a=0.3, b=1, c=0.01, J=2 and $v_0=1$