AFFECT AND ACHIEVEMENT: CREATING AN OPTIMAL LEARNING EXPERIENCE IN MATHEMATICS

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DEDICATION

This thesis is dedicated to my husband, Kevin Skousen, for always believing in me, and to our 5 children for their patience, especially with philosophical discussion about mathematics at the dinner table.

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ABSTRACT

ELIZABETH SKOUSEN

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This paper is an analysis of cross-curricular studies about motivation, affect, and engagement in a classroom setting. I further determined appropriate tools for measuring student engagement and affect. Student self-efficacy is a determining factor in motivation and engagement in the classroom. Problem-based learning, mastery learning, and student self-assessment are three instructional methods of particular interest in increasing student engagement and motivation. I considered each of these instructional methods in turn as ways to enhance student self-efficacy and positive affect and conduct a statistical analysis on the correlations between these measures.

Problem-based learning, mastery learning, and student self-assessment are positively correlated with student affect, motivation, and engagement, which contribute to student achievement and future learning of mathematics.

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CHAPTER I

A HISTORICAL PERSPECTIVE ON PROBLEM SOLVING

At its heart, mathematics is the science of finding and making use of patterns. As new mathematical theories arise, more applications of those theories come into existence, and in turn, the need for an application drives creation of mathematical theory. As theory develops, new opportunities for application arise, and as advances in science, engineering, and technology occur, so do new developments in mathematical theory. In some cases, scientific advancements lead to the development of an entirely new branch of mathematics.

Astronomy and Trigonometry

For example, the development of the field of trigonometry was largely dependent on advancements in astronomy, especially in spherical geometry (Hunt, 2000). Hipparchus, in his work with the astronomical signs of the zodiac, created or derived a table of trigonometic ratios. In order to do so, Hipparchus considered a triangle as being inscribed on a circle, with the sides of the triangle therefore also chords of the circle. He applied the Babylonian principle of dividing a circle into 360 degrees, and from there created generalized ratios of the sides of a triangle. Menalaus contributed identities for spherical triangles, given by intersections of great circles. This work was developed because of a need to explain spherical phenomena in the night sky. Ptolemy expanded on this work in his Almagest to create a table of chords equivalent to a table of sines, as well as using theorems and identities equivalent to those used today (Elert, 1994).

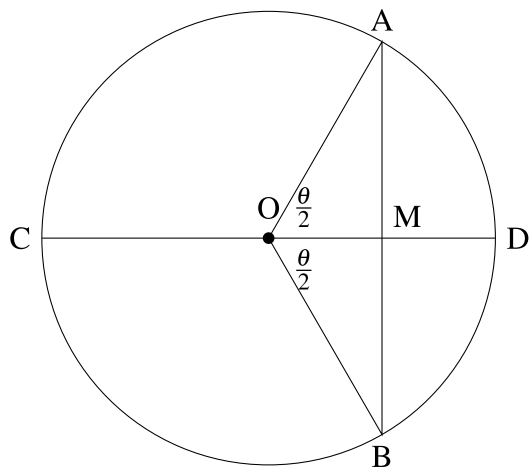

Figure 1. A central angle inscribed in a circle

Ptolemy created a table of chords for circles like those in Figure 1, which corresponds to the table of sines to the half-degree, given that for this circle,

$$\begin{aligned}\sin\frac{\theta}{2} &= \frac{AM}{OA} \\ &= \frac{2AM}{2OA} \\ &= \frac{AB}{CD}\end{aligned}$$

so the sine of $\frac{\theta}{2}$ is the ratio of the chord length to the diameter of the circle.

Problem Solving in Calculus

The field called calculus today is an amalgam of several theoretical branches of mathematics, many of which arose from the problems that scientists and engineers needed to solve. Early development of calculus involved the estimation of the infinite by the method of exhaustion, which would be recognizable today as the study of limits. Archimedes, for example, used the method of exhaustion to estimate the volume and surface area of spheres and cones, as well as the area of an ellipse. He further uses this method in estimating volumes of segments of paraboloids and hyperboloids of revolution (O'Connor & Robertson, 1996). Figure 2 shows Archimedes' diagram for approximating the area of a circle using this method.

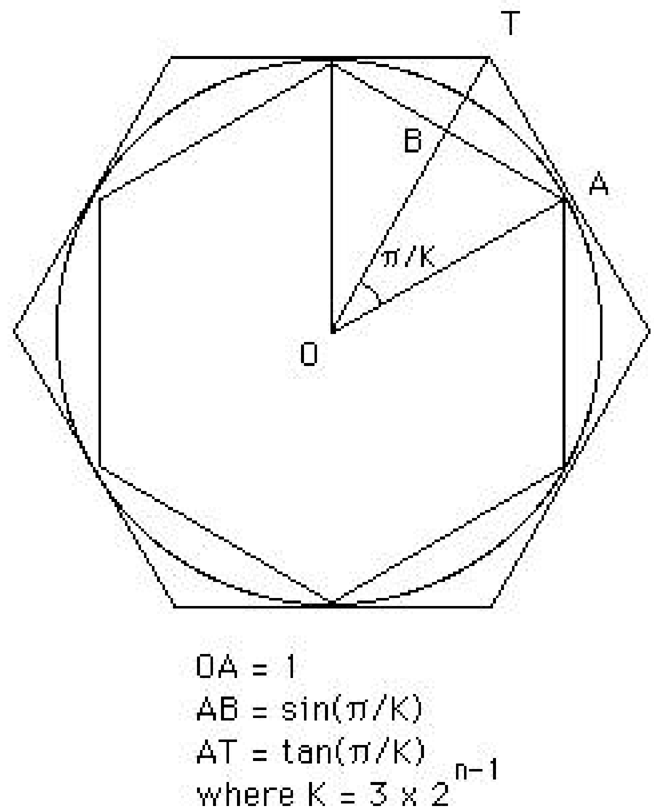

Figure 2. Archimedes' diagram for approximating the area of a circle

After this time period, the study of methods of exhaustion, or limits, did not make much progress until the 16th century. At this point, an interest in mechanics began to drive mathematicians to study problems involving centers of gravity. Luca Valerio attempted a type of integration of parabolas to attack this problem, while Kepler's interest in planetary motion involved approximating the area of sectors using smaller and smaller sectors that approached lines (O'Connor & Robertson, 1996).

Valerio's work, *De Centro Gravitas*, uses this Archimedean method to calculate the centers of gravity of various solids of rotation. One interesting lemma of this work coincides with Lemma IV of Book I of Newton's *Principia*. This lemma shows that for $\lim x = a$ and $\lim y = b$, if $\frac{x}{y} = c$, where $c \in \mathbb{R}$, then

$$\frac{a}{b} = \frac{\lim x}{\lim y} = \lim \frac{x}{y} = c.$$

Cavalieri theorized principles of volume being made of smaller and smaller sections, leading to the 'Cavalieri principle' taught in high schools; namely, that the volume of a three-dimensional object is not dependent on its slant. In other words, for two solids of the same height, if every corresponding cross-section has equal area, then the solids will have equal volume. Cavalieri's work *Specchio ustorio*, printed in 1632, is concerned with reflecting mirrors for the purpose of resolving the age-long dispute of how Archimedes allegedly burned the Roman fleet that was besieging Syracuse in 212 B.C. The book further considers properties of conic sections, reflection of light, sound, and heat (Ariotti, 1975). This work may be evidence that Cavalieri invented the reflecting telescope before even Isaac Newton or James Gregory, as he posited that combining convex mirrors with a convex lens would produce a telescope.

Mathematicians interested in these developments began to create proofs and theorems that developed into what is known today as differentiation and antidifferentiation, or integration. Newton became interested in understanding the relationship between position, velocity, and acceleration, and began to formalize the theories using what he termed fluxions and infinite series.

Newton's 'method of fluxions' is based on the idea that the integration of a function and differentiating a function are inverse procedures. Using differentiation as the basic procedure, Newton showed analytical methods unifying separate techniques that solved previously unrelated problems. These seemingly unrelated problems were finding areas, tangents, the lengths of curves, and the maxima and minima of functions. The unification of these ideas as inverse procedures of differentiation led to the idea used today of antidifferentiation (O'Connor & Robertson, 2000).

Many of Newton's contributions to mathematics arose from observing natural phenomena, such as motion of planetary bodies, projectiles, and the properties of light. Newton used these observations to look for patterns and find an underlying mathematical theory connecting these seemingly disparate sciences. Some of these phenomena that had been previously unrelated include the orbit of comets, variation of tides, precession of the earth's axis, and the way that the gravity of the Sun impacts the motion of the moon (O'Connor & Robertson, 2000). Liebnitz further formalized infinitely successive differences and the ratio $\frac{dy}{dx}$, which notation is still commonly used in calculus today.

Problem Solving and Abstract Mathematics

Sir Isaac Newton, among others, noticed properties in the physical world that could not be explained with current mathematical knowledge and theory. As a result, Newton began to study the mathematical patterns of change in the physical world, especially as it applies to gravity. The need to solve these kinds of problems led to the development of the field of Calculus.

When this field of mathematics was not sufficient to explain the way time and gravity act together, an entirely new field of mathematics was born. Einstein began solving problems about simultaneity, and as a result special relativity and later general relativity opened up new branches of mathematics.

When observations and applications contradicted current mathematical theory, it became obvious that mathematical theory was what needed to expand. Some historical mathematicians came to make great contributions by wanting to solve more abstract problems. Euler, an incredibly prolific mathematician, studied previous works and sought to determine whether theories were always or only sometimes true.

In one notable example, Euler was asked in 1729 whether Fermat's conjecture that the numbers $2^n + 1$ were always prime if n is a power of 2. Euler showed by counterexample for $n = 32$ that the number $2^{32} = 4294967297$ is not prime, since 4294967297 is divisible by 641.

Problem Solving and the Future of Mathematics

Leonhard Euler had projects in multiple related subjects, including cartography, science education, magnetism, fire engines, machines, and ship building. His mathematical research included number theory, analysis and calculus, and differential equations, which he viewed as related to his work in rational mechanics (Cameron, 1987).

As Euler and other mathematicians throughout history showed, the development of mathematics and applications or problem solving go hand-in-hand. Solving new problems today expands mathematics as well. Some questions driving current and future mathematical theory are: Is it possible to untangle DNA? Why is it so difficult to predict the weather? What is the most secure way to encrypt data? Answering these questions has led to developments in graph theory, knot theory, and chaos theory. These questions of scientific inquiry are just as much mathematical inquiry. New fields of mathematics and branches of study have resulted from trying to answer these questions and many others.

This paper discusses why problem solving is integral to learning mathematics and how utilizing problem solving in the mathematics classroom changes the outcome of mathematics students.

CHAPTER II

LITERATURE REVIEW

Affect and Cognition

The way we feel influences the way we learn. In psychology, this is called affect, and positive and negative affect correlate with positive and negative mood and emotions (*Dictionary of Psychology*, 2020).

For example, anxiety, which is a negative affective experience, is seen in individuals who also experience academic difficulties, such as lower GPA, lower standardized test scores, and decrease in graduation rates (Moran, 2016). Further, anxiety is related to a decrease in working memory capacity, which includes the cognitive abilities to "select relevant information, resist distraction and maintain access to relevant information outside of immediate awareness" (Moran, 2016, p. 843). These cognitive tasks are directly related to the capability to solve problems, recall information, and perform on assessments. Other highly negative emotional contexts also impair working memory capacity in groups of students and community subjects alike (Schweizer & Dalgleish, 2016). Math anxiety is one example of a negative affective state about mathematics. Studies show that math anxiety is moderately linked to poor math performance on tests, independently of text anxiety (Dew, Galassi, & Galassi, 1984).

Motivation

Motivation is what moves or drives a behavior. In mathematics classes, that motivation looks like student engagement and participation in class. One theory of motivation, causal attribution theory, describes attributes such as success or failure that determine whether a student wants to continue doing a particular task and what causes that success and failure. There are internal causes, such as ability, effort, and mood, and then there are external causes such as difficulty of the task, luck or coincidence, and help from others. In the case of the student, whether these internal and external causes exist can determine how motivated the student is for success and failure (Williams & Ivey, 2001). In the case of Bryan, a student observed by Williams and Ivey (2001), the researchers noted that he did not care about success or failure, and therefore, these causal attributes were insufficient to describe his motivation and behaviors.

Another theory of motivation is self-efficacy theory, which has to do with whether an individual judges themselves as capable of doing a task. In particular, an individual with high self-efficacy can organize and implement actions in situations that might be new or unpredictable. If a student predicts that they do not have self-efficacy for a task, then they would seek to avoid that task, whereas a student with positive self-efficacy would participate or engage more readily. This is evident when mathematics students make comments such as 'I'm no good at math' and do not attempt problems or tasks. In Bryan's case, however, he did say "Well, I'm good at it [mathematics] in some parts, you know. In some respects, I'm pretty good at it I think" (Williams & Ivey, 2001, p. 89). He did perceive himself as capable of doing the task, but he was not motivated to attempt it, which leads to the conclusion that self-efficacy theory is not sufficient to describe his behavior.

Perceived usefulness is a theory that when an individual perceives a task as useful or necessary, they are more likely to be motivated to do the task. When a student says, 'when am I ever going to use this?' or 'why do I need to know this?' they are expressing motivation due to perceived usefulness. For these individuals, if they believe that mathematics is likely to be useful, then it should encourage more participation in mathematics tasks. Again, this was not the case for Bryan. He did express that the tasks they were instructed to do were likely to be useful in his future, but he did not engage with the tasks (Williams & Ivey, 2001).

The theory of goal orientation includes learning goals or performance goals and suggests that increased effort results in greater competence. A person who is learning goal oriented believes that ability or intelligence can be increased through effort, while a person who is performance goal oriented believes that intelligence and ability are fixed. Learning goal theory suggests that a student with this mindset will seek challenges and mastery, since they believe that continued effort will lead to success. Current work in this area encourages teachers and administrators to create and support a "growth mindset" among students.

For a student who still is not engaging in mathematics, there may still be a missing piece—affect, or emotional connection to the subject. In practice, the student may say 'I know how to do it, but I just don't like it' or 'I know if I put in time I would do better, but I just don't want to.' These types of statements indicate a negative affect or emotional connection with the subject.

The model of affect and emotional experience indicates that a strong emotional arousal can lead to a reduced capacity for problem-solving (Malmivuori, 1989; McLeod, 1989). Experiencing a strong negative affect towards mathematics actually reduces capacity for problem-solving, a skill that forms the basis for mathematics achievement. When students experience negative emotion, including anxiety, frustration, and low self-esteem, in relation to mathematics, they are less able to perform the tasks that they have the negative affect towards.

In the case study performed by Williams and Ivey (2001), the researchers noticed that Bryan became much more engaged and participated in the mathematics activities on specific occasions where he had an opportunity to insert himself–in particular when he saw that his contributions were valuable and was able to put something of himself into the lesson for the day. At one point, "Bryan was truly engaged in the classroom work, even excited about what was happening" (Williams & Ivey, 2001, p. 95). The authors further noted that this occasion was the first time students had been asked to justify their reasoning for their solutions. They describe that Bryan "was animated and tried hard to express his reasons for doing things the way he did. Bryan saw, for the first time all year, an opportunity to put himself into the mathematics" (Williams & Ivey, 2001, p. 95).

Positive emotions about mathematics increase student creativity, capacity for problem solving, and comprehension of material. Having a positive experience during mathematics learning also contributes to motivation to engage and learn in the future (Csikszentmihalyi, Rathunde, & Whalen, 1997; Rathunde & Csikszentmihalyi, 2005). I consider now several instructional methods that increase positive affect, engagement, and achievement.

Autonomy, competence, and relatedness are three important elements related to the quality of motivation (Ryan & Deci, 2000). Each of these elements in mathematics education can be addressed with the instructional tools of problem-based learning, mastery learning, and self-assessment. Problem-based learning (PBL), which utilizes student problem-solving as a form of instruction, is historically relevant as a method that impacted the growth of mathematics. It is furthermore a tool for the motivation and engagement of the student in a classroom setting. PBL is a method of instruction that allows students to bring their past experience to the table in ways that are seen as valuable. It is a type of student-centered curriculum and instruction that gives students like Bryan an opportunity to connect with the material which may in turn increase motivation to engage with the class. Problem-based learning meets these motivational needs of relatedness and competence, and a true implementation of PBL also allows for autonomy.

Defining and Measuring Engagement

Several studies define engagement as composed of behaviors, emotions, and cognitions (Fredricks, Blumenfeld, & Paris, 2004; Jimerson, Campos, & Greif, 2003). Educational research is concerned with the actions and behaviors that students have in school and direct towards learning. The behavioral engagement of students includes involvement and participation in academic tasks, which are evidenced by student attendance, completion of work, effort, and concentration (Finn, 1993; Finn, Pannozzo, & Voelkl, 1995). Emotional engagement refers to student affect, shown by reactions and identifications that students make in the school environment (Skinner & Belmont, 1993). A student's cognitive engagement can be measured by their strategic approach to learning (Fredricks et al., 2004).

When measuring student engagement, it is important to have a repeatable, reliable scale for tests. Wang and Holcombe (2010) developed the behavioral engagement related to instruction (BERI) protocol, which was shown to be repeatable and reliable between researchers and participants (Lane & Harris, 2015). This protocol takes into consideration behaviors that are evidence of behavioral, emotional, and cognitive engagement in the classroom.

During classroom observation, researchers found that supporting student autonomy did lead to increased student engagement. "When other predictors are controlled, students who engage in schoolwork out of inherent interest or internalized values report more involvement, participation, curiosity, and enjoyment. Further, autonomous motivation is associated with less avoiding and boredom" (Shih, 2008, p.329). Finding instructional methods that increase student autonomy can lead to increased engagement and positive affect, which will in turn create opportunities for greater future learning.

One such instructional tool is PBL. Learning is more likely to occur when learners enjoy themselves, experience self-efficacy, and have positive self-worth (Kaufman & Mann, 2001; Petranek, 1994). Therefore, a successful instructional method will help students build these attributes. This suggests that "fostering positive emotions among students can provide the basis for meaningful learning. In this sense, the emotionality of learners in PBL is of great importance" (Takahashi & Saito, 2013, p. 703).

CHAPTER III

PROBLEM SOLVING IN MATHEMATICS EDUCATION

Problem-Based Learning

I now consider educational tools that incorporate student choice and agency in the learning process. One such tool, problem-based pedagogy, values student input and bases its instructional model around the idea that previous student experience and skills enhance the learning of new skills. For this reason, a task chosen in the problem-based classroom will be accessible to all students and increase level of difficulty in a scaffolded manner.

The Center for Gifted Education at the College of William and Mary and the Illinois Math and Science Academy focus on the following goals for problem-based learning:

- Develop and retain extensive content base with deep conceptual understanding
- Develop and use flexible problem-solving skills
- Develop and enhance self-directed learning strategies
- Develop lifelong engaged learners (Burruss, 1999).

Of particular interest are the goals of enhancing self-directed learning, which is tied to affect, and the development of engaged, motivated learners.

PBL uses contextual problems in a classroom setting to highlight student experience and prior knowledge. The setting combines problem-based curriculum with a pedagogical structure that values student voice (Schettino, 2011). Student contribution is specifically valued and encouraged. This type of learning environment and instructional approach provide the opportunity for self-insertion that Bryan (Williams & Ivey, 2001) and other students need in order to have a positive affect and engagement toward math.

Current good practice for a problem-based mathematics classroom includes the following features:

- Clarify the task at hand. This includes "task criteria," referencing the process, or how to accomplish the task, as well as the "quality criteria," or what constitutes doing the task well.
- Understanding that criteria may be dependent on other factors, and some criteria outweigh others in certain circumstances.
- Meta-cognition, or thinking about how to think about a task, which can aid in the transfer of criteria to other situations (Torrance, 2012).

These features include the need to create clear objectives or 'task criteria,' clear knowledge of how to determine whether a student has accomplished the task or 'quality criteria,' and the ability to justify or provide evidence for the task. Using mastery learning and assessment in the classroom provides these features while also increasing student self-efficacy. Effective problem solving creates productive struggle in students. Unlike direct instruction, students are given a problem that they must grapple with, developing their critical thinking and problem-solving skills. When errors during the thinking process still create opportunities for discussion and learning, this struggle is seen as productive. Table 1 describes some characteristics of productive struggle in a problem-based classroom.

Mastery Learning and Assessment

One major shift in the theory of education came with the development of mastery learning, also referred to as Learning For Mastery (LFM). The shift came when John Carroll noted in 1963 that "students differ in the amount of time they need to learn material to a set criterion, and thus the central variable that must be changed is time" (Diegelman-Parente, 2011, p. 50). Later studies showed that the ratio of learning times for the slowest

5% of learners and the fastest 5% of learners is about 5:1, since they take up to 5 times as long to reach the same learning criteria and objectives (Bloom, 1974).

Table 1
Describing Productive Struggle

Productive Struggle	NOT Productive Struggle
Students engaging with problems without readily apparent solutions by using existing understandings	Students waiting for information or instruction so they can memorize or practice it
Students persevering in solving problems and making sense of the mathematics	Students giving up and feeling despair because the mathematics does not make sense
Students solving problems while developing understanding of the key mathematical ideas that are within reach	Students experiencing frustration with extreme challenge and excessive difficulty

adapted from Baker, Jessup, Jacobs, Empson, and Case (2020)

In mastery learning, researchers and educators seek to adjust the classroom learning to give students the needed time to reach the criteria rather than a uniform experience. Students are responsible for meeting criteria individually, and the course is structured to give extra support and feedback to those who need more time and enrichment opportunities for those who need less. In studies in which this model has been implemented in the classroom, adjusting for time in the mastery class does in fact help students reach achievement levels. Bloom (1974, 1978) notes that about 80% of students in the mastery learning class environment reach the levels of achievement that only the top 20% reach in a conventional learning environment.

We can compare mastery-based instruction with traditional instruction. In a traditional classroom, students are given instruction about a new topic. They then complete independent work, including homework, projects, and exams, which are graded by the instructor. The instructor then determines whether the student receives a passing or failing grade on the work. In some cases, the instructor allows failing students to receive remediation for failed work. Then students proceed to the next unit of instruction (see Figure 3).

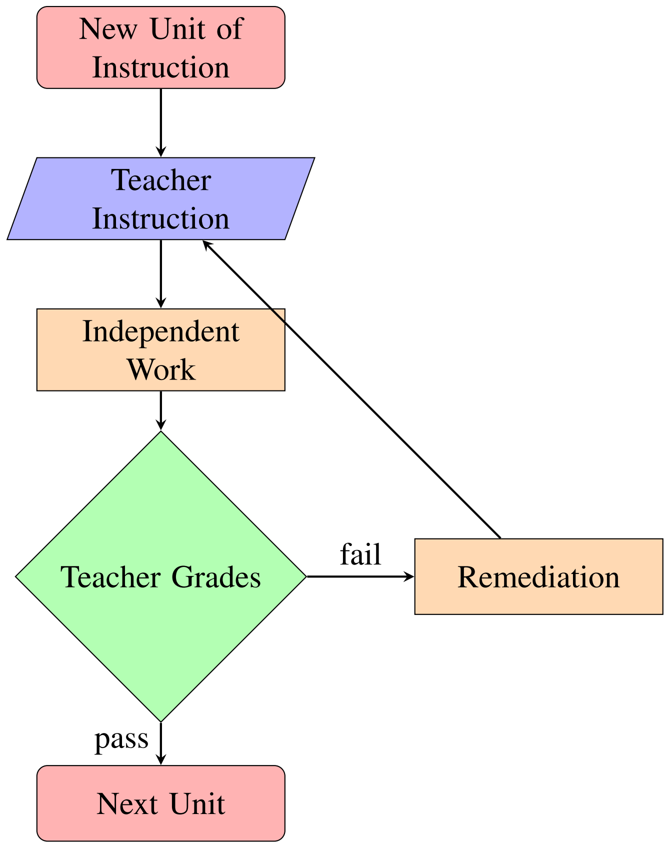

Figure 3. Learning process in a traditional classroom

In a mastery-based classroom, students receive goals and objectives before the unit begins, and they have a clear path to what mastery of the subject (whether it is a standard, topic, or unit) entails. Students engage with the material and have opportunities to show mastery of individual objectives as well as the larger unit or subject goals. The cycle does not end with assessment or scoring, however. Every student compares their work with the unit goals and objectives and determines whether they have demonstrated mastery on each.

For students who show mastery on only some objectives or goals, they have the opportunity to correct their work and show that they have now achieved mastery. For students who have successfully demonstrated mastery on each goal or objective, they have an opportunity for enrichment to deepen their understanding of the topic. Then, after all students have had an opportunity for correction and enrichment, a new unit of study with corresponding goals and objectives begins. In this way, mastery learning is a cyclical, student-driven instruction style, as opposed to the linear, teacher-driven model of traditional instruction.

Hartley (1974) outlines some specific needs that a mastery-based classroom should include:

- The learner should be given some clear idea of where he is going, i.e., the terminal behavior.
- The instruction leading to this behavior must be sequenced into small steps.
- The learner should work on each step alone and at his own pace.
- At each step, the learner should be encouraged to actively respond.
- The learner should receive immediate knowledge of results concerning the correctness or appropriateness of these responses. (p. 279)

A unit of instruction should have goals or objectives, available to the student at the beginning of study. In the case of the US K-12 school system, states have mathematical standards that must be met. Ideally, curriculum would adhere to the standards and provide objectives or learning goals that break the path to meeting each standard into clear, manageable steps. For private schools, advanced coursework, and higher education, there are not necessarily specific standards developed. In this case, curriculum developers or instructors should carefully consider the overarching goals for each unit of instruction and identify learning goals and objectives that will help students achieve the unit goals.

Students should participate in learning activities that show progress for each objective. This includes in-class participation, problem-solving tasks, group discussions, reflections, and independent work. Instructors must provide immediate feedback on student work, so that students can correct errors and deepen understanding. Mastery learning includes a cycle of student work, feedback, and remediation or enrichment. This provides the necessary time for mastery of each objective as well as the standard or unit goals (See Figure 4).

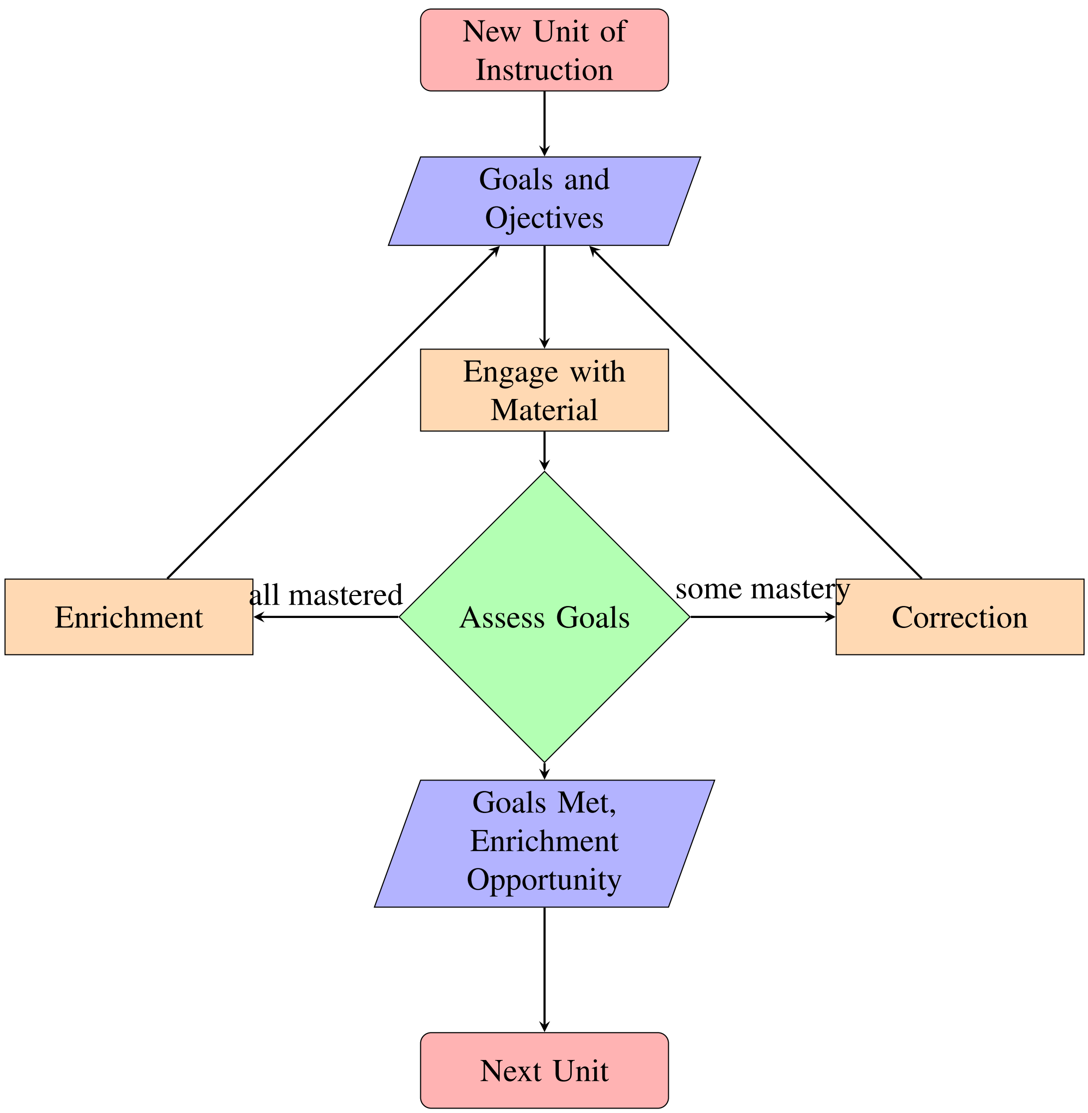

Figure 4. Learning process in a mastery-based classroom

As students begin to take responsibility for their education several aspects should be considered:

- Student willingness to engage
- Student effort and perseverance
- Student aptitude
- Opportunity for the student to learn
- Quality of education and instruction

"The degree of school learning of a given subject depended on the student's perseverance or his opportunity to learn, relative to his aptitude for the subject, the quality of his instruction, and his ability to understand this instruction" (Block & Burns, 1976, p. 5). Mastery learning affords several of these aspects. Students are rewarded for increased effort and perseverance through feedback, revision, correction, and enrichment. Students are given more opportunity for learning as they receive the time they need for each objective.

When assessing student mastery, it is important to use tests that reference the criteria or objectives rather than population norms. This reinforces to students that attaining goals is more important in their education than student comparison (Diegelman-Parente, 2011). That is, to motivate students to take the time they need to achieve mastery, assessments as well as tasks should be based on objectives and goals rather than comparisons of how well a student tests relative to other students.

A mastery-based assessment has the following qualities:

- Concrete and clear goals. Goals should clearly help learners understand the required task and the procedures to follow in learning it.

- Small, manageable objectives. These objectives should help students reach the unit goals and be tested at the end of each unit.

- Effective feedback. Teachers should identify particular errors and difficulties on the test so that students can review and correct these errors.

- Sufficient time for all students. Individuals require different amounts of time to learn, so teachers should consider ways to alter the amount of time available to test for some students.

- Alternate opportunities. Some types of alternate learning opportunities might be projects, papers, and presentations. These alternate opportunities should still assess whether a student meets the same goals and objectives of the unit.

- Review and revise. "Student effort is increased when small groups of two or three students meet regularly for as long as an hour to review their test results and to help one another overcome the difficulties identified by means of the test" (McNeil, 1969, p. 308).

Interestingly, students who have been instructed using mastery-based methods fare better even on conventional assessments and achievement tests; in "97 comparisons of average achievement test scores, comparisons involving various types and numbers of students and various subject matter areas, mastery-taught students scored higher than non-mastery-taught students 89% of the time, and significantly higher 61% of the time" (Block & Burns, 1976, p. 19). Overall, students taught using principles of learning for mastery score higher than traditional students. Additionally, the groups of students in mastery-based learning groups have less variability in their achievement as well. In fact, "in 80 comparisons of the variance in achievement test scores of mastery-taught and non-mastery-taught students, mastery students exhibited less variability 74% of the time" (Block & Burns, 1976, p. 21).

This indicates that the additional opportunities for learning and increased time to master objectives and goals may close the learning gap between the highest and lowest 5% of students.

Mastery learning impacts more than student achievement or outcome. Students in a mastery learning classroom reported a "70% satisfaction score with the process, and 80% reported they would choose mastery learning for a subsequent course, similar to findings reported elsewhere" (Diegelman-Parente, 2011, p. 55). This implies that student affect and future learning are positively impacted in the mastery learning classroom.

Student Self-Assessment

PBL and mastery learning both increase student self-efficacy in the classroom. These strategies, along with self-regulatory strategies that help students monitor, regulate, and control their cognition, motivation, and behavior, are the strategies that will lead to the most successful academic outcomes (Bercher, 2012; Pintrich & de Groot, 1990). In the classroom, these self-regulatory strategies are most evident in student self-assessment.

Some benefits of accurate self-assessment include using feedback effectively, improving time management skills, and setting more appropriate goals (Hacker, Bol, Horgan, & Rakow, 2000). This leads to an increase in self-efficacy and self-regulated learning skills, which helps students be more successful in the classroom (Bercher, 2012). In classroom research, self-efficacy had a statistically significant correlation with test scores or achievement in mathematics (Malmivuori, 2006). Interestingly, there was also a strong correlation with enjoying and liking mathematics.

Using learning goals in the classroom and giving students frequent opportunity to self-assess their mastery toward goals is designed to encourage students to take responsibility for their progress, fostering a sense of self-efficacy. Students should have the opportunity to self-evaluate progress on a daily basis, both formatively and summatively, as well as cumulatively at the end of a unit or term. Assessment and feedback is a process involving students, teachers, and peers, and should be a reflection of information from the demonstrations, observations, and dialogue occurring daily (Klenowski, 2009).

Assessment, especially self-assessment, does not only look like an exam at the end of a unit of content. Rather, assessment involves self, peer, and teacher feedback at every stage of the learning process. Self determination theory describes both autonomous and controlled motivation as aspects that do motivate a learner or participant, but they have different effects and outcomes. Intrinsic motivation, or motivation to engage because of interest, is most like autonomous motivation, or motivation due to alignment with goals, values, and regulations. These are very dissimilar from controlled motivation, based on contingencies of rewards and punishment (Gagné & Deci, 2005). For this reason, student self-assessment in a mastery-based classroom would have the most beneficial effects on student motivation and engagement.

Student self-assessment also gives students an opportunity to feel that they have responsibility for their own learning. They can predict and monitor their own progress, combining the need for autonomy and competency. Encouraging a learning-goal orientation may increase motivation and introduce a feeling of competence internally for students, helping them become more willing to engage with material. Mastery learning and assessment helps students be more engaged by increasing their motivation and helping them become more learning-goal oriented. Areas for future research would include how the interaction of PBL, mastery learning, and student self-assessment together impact student motivation, affect, and achievement. Research across disciplines shows that motivation, affect, and engagement are positively correlated with problem-based learning, mastery learning, and student self-assessment in the classroom. A further analysis is needed of how these instructional tools interrelate and support one another in the classroom to further enhance student learning.

CHAPTER IV

PROBLEM-SOLVING TASKS

Rotations

How does rotation affect an object? I will consider how problem solving and rotation look during multiple stages of mathematical development by examining several problem-based tasks.

Elementary

For the elementary task (See Appendix B), students are given shapes partitioned into fourths. They must explain how they know whether each partition is the same size. Without explicitly being a task about how rotations act in $\mathbb{R}^3$, which is beyond the developmental levels of a typical first grader, students engage in the main problem-solving skills to reason about shapes in Euclidean space.

Consider these two partitioned "cake" pieces in Figure 5 that students might compare.

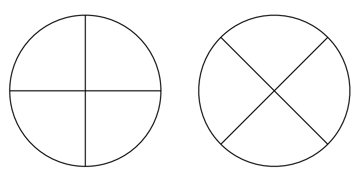

Figure 5. Two "cakes" students may compare during the task

Some of the problem-based learning features students encounter in this task include:

- Decide what determines equal in this context.
- Consider how to know whether two shapes are equal.
- Decide whether changes in orientation affect equality.

- Compare partitions within shapes and between shapes.
- Justify reasoning about equality and partitions.

This task supports student reasoning about rotations as they consider whether two shapes, identical except for orientation, are partitioned into equal parts. They bring personal experience, that objects do not change size or shape in Euclidean space (the real space they inhabit) when they are turned or rotated. They use this experiential knowledge to make conclusions about partitioning, and this knowledge will be further expanded in future tasks.

Secondary

In this task, students in high school geometry are asked to rotate line segments about a point and make conjectures about the results (see Appendix B).

Consider some of the conjectures that students might make:

- Rotations preserve the distance to the center of rotation.
- The angle bisector of angle *A* is also the perpendicular bisector of the base (*BC*) of an isosceles triangle.
- The two base angles (angle *B* and angle *C*) of an isosceles triangle are congruent.
- An isosceles triangle where angle *A* is 60 degrees is also equilateral.

Students are using some existing knowledge, such as how angle bisectors and perpendicular bisectors are constructed, to create new conjectures; namely, that the angle bisector of the apex of an isosceles triangle is also the perpendicular bisector of its base as well as further explore congruence with rotations and triangles. The major takeaway after some related tasks is that rotations are a rigid transformation in Euclidean space, meaning that the image of a figure under rotation is congruent to its preimage.

Graduate

Problem solving at the graduate level can take on a much more relaxed structure. For example, an open-ended project allows the student to engage in self-directed problem-solving while researching.

For this task, the student chooses to describe projective change in $\mathbb{P}^3$ using rotations.

Prior learning

To begin the problem-solving approach, the student considers existing knowledge about affine change.

Garrity et al. (2010, p. 11) gives a general formula for real affine change in two dimensions:

$$u = ax + by + e$$

$$v = cx + dy + f,$$

where $a, b, c, d, e, f \in \mathbb{R}$ and

$$ad - bc \neq 0.$$

This can be rewritten in matrix language as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix},$$

where $a, b, c, d, e, f \in \mathbb{R}$, and

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

Furthermore, where A represents the symmetric matrix associated with a homogenous equation, and B represents the matrix associated with a change of coordinates or transformation, the transformation will map the original equation to the new homogenous equation associated with the matrix N, where

$$N = (B^{-1})^T A B^{-1}.$$

According to Emery (2015, p. 17), the matrix form of coordinate change can be used to find the equation for a rotation about the x-axis by θ radians in $\mathbb{C}^3$. The transformation matrix B is represented by

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

As a geometric representation, the mapping from $\mathbb{C}^3 - (0,0,0)$ to $\mathbb{P}^2$ helps to visualize what projective transformation does to a polynomial. Consider again the circle centered at the origin with radius equal to 1. The equation of this conic section is $x^2 + y^2 = 1$. In order to map it to the projective plane, the equation is homogenized as $x^2 + y^2 = z^2$. This looks like the quadratic surface in $\mathbb{C}^3$ known as a double-napped cone. The original equation occurs when $z = 1$, or when the intersection of the double-napped cone and the plane parallel to the *xy*-axis is taken at $z = 1$, seen in Figure 6.

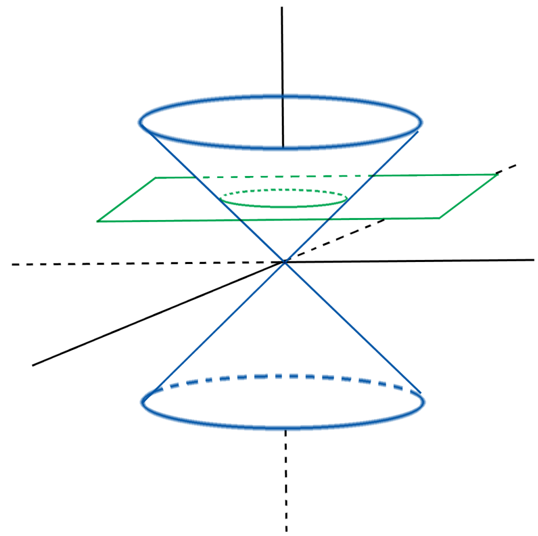

Figure 6. The elliptic cone $x^2 + y^2 = z^2$ and plane $z = 1$

The illustration given above demonstrates the well-known principle that the intersection of a double-napped cone and a given plane is another conic section. A geometric representation of projective change is to perform transformations on the homogenous formula, then consider the resulting plane intersection at $z = 1$. For example, the unit circle can be transformed projectively into a parabola. The homogenous equation $x^2 + y^2 = z^2$ represents a cone in $\mathbb{C}^3 - (0,0,0)$. Rotating the cone counterclockwise by $\frac{\pi}{4}$ radians about the x-axis and intersecting this new cone with the plane at $z = 1$ results in the equation $x^2 - 2y = 0$, which is a parabola (see Figure 7).

The unit circle can also be transformed projectively into a hyperbola by rotating the cone counterclockwise by $\frac{\pi}{2}$ radians about the x-axis and intersecting this new cone with the plane at $z = 1$, shown in Figure 8. The resulting equation will be $y^2 - x^2 = 1$, a hyperbola which opens about the y-axis.

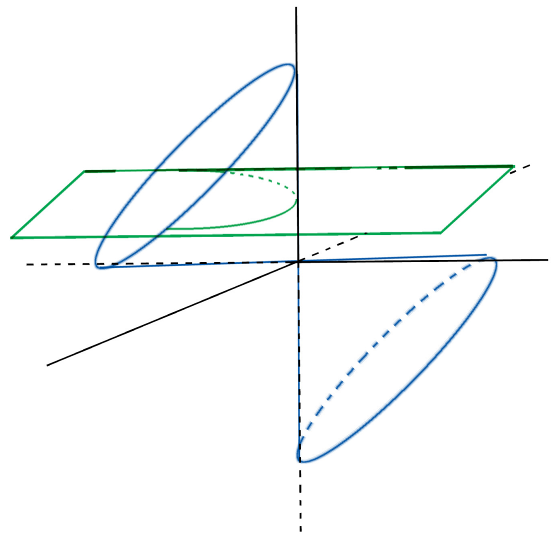

Figure 7. The elliptic cone after $\frac{\pi}{4}$ rotation and the plane $z = 1$

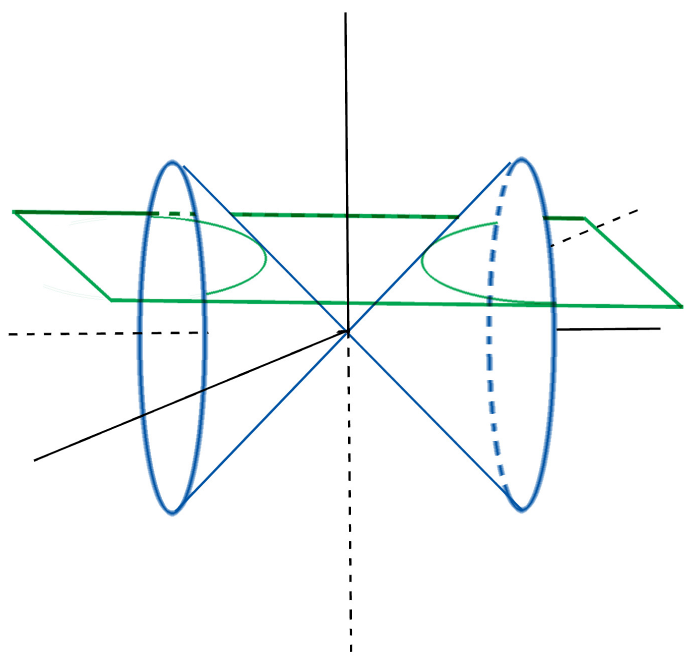

Figure 8. The elliptic cone after $\frac{\pi}{2}$ rotation and the plane $z = 1$

This projective representation can also be used to transform from a circle to an ellipse by rotating the cone counterclockwise by $\frac{\pi}{12}$ radians about the$x$-axis and intersecting this new cone with the plane at $z = 1$, shown in Figure 9. The resulting equation is $2x^2 + \sqrt{3}y^2 - 2y - \sqrt{3} = 0$.

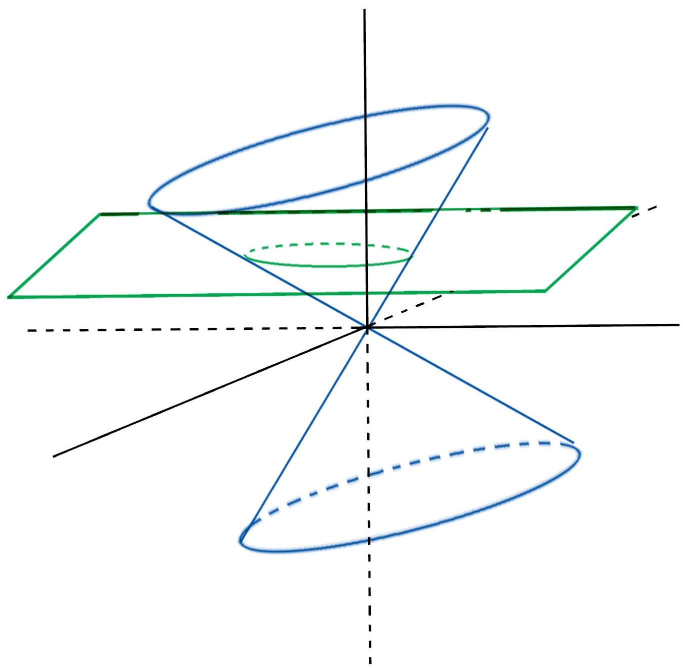

Figure 9. The elliptic cone after $\frac{\pi}{12}$ rotation and the plane z=1

Applying new learning

A projective change of coordinates can change one quadric surface into another type. As an example, consider the unit sphere in three dimensions, given by $x^2 + y^2 + z^2 = 1$. This equation can be homogenized in the projective plane as $x^2 + y^2 + z^2 = w^2$, which can be represented by the symmetric matrix A.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

To apply the process discussed in previous sections, a transformation matrix is chosen that represents a rotation in two of the four dimensions, one of which is the projective variable w. Therefore, the transformation matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

represents a rotation of $\frac{\pi}{2}$ in the z and w dimensions. The resulting matrix

$$M = (C^{-1})^T A C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

represents the homogenous equation $x^2 + y^2 - z^2 + w^2 = 0$. When dehomogenized again at $w = 1$, the resulting equation $x^2 + y^2 - z^2 + 1 = 0$ is the equation for a hyperboloid of two sheets in $\mathbb{C}^3$. Shown here in this example, under a projective change of coordinates, a spheroid is isomorphic to a hyperboloid of two sheets. Thus, the process for a projective change from one type of conic section to another can be generalized to more dimensions. For further examples, it is only left to choose appropriate transformation matrices.

Bringing it Together

In this example, the task is open-ended, and the student displays the features of PBL by creating bridges between prior knowledge and the new mathematical principles being explored. A problem-based approach to understanding rotations and other mathematical principles is possible at every level of mathematics.

In these cases, students consider other related mathematical principles than just the idea of rotating an object. For example, in the Grade 1 task, students consider what ‘equal’ means in the context of equal partitions. They do not yet have the language to discuss congruence, or the mathematical experience to compare equal areas. This meta-cognition about equality of shapes continues as students use understandings about rotations to create conjectures and prove theorems in high school geometry. In upper-level mathematics, students consider different types of equality, including congruence, affine change, and isomorphisms.

Optimization

Optimization is the study of maximizing or minimizing the outcome of some function(s) with respect to constraints. At early stages, this looks like questions about what is the best outcome, or the most, or the least in regards to some descriptive constraints. At higher stages, I consider statistical methods of convex optimization.

Preschool

Tasks for early learners that focus on developmental problem-solving skills can build a fundamental support for later discovery of optimization and constraints. One task an early learner can do is build a tower. First, give the learner access to a limited amount of bricks or blocks. In this illustration (see Figure 10), 20 bricks are provided. Then have the learner construct the tallest tower they can with the bricks, the widest tower, and a strong tower (see Figures 11 and 12).

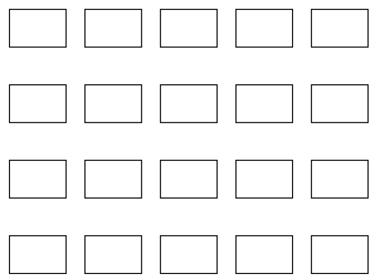

Figure 10. 20 bricks to complete the task

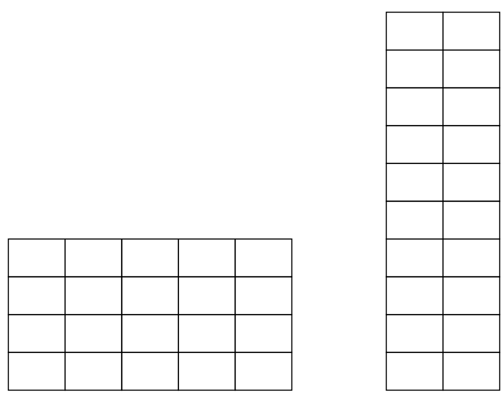

Figure 11. Sample towers with strong bases

The target or goal, a tall tower, a wide tower, or a stable tower, is being maximized subject to the constraint of a limited number of bricks. In this case, what the student uses as criteria to determine what makes a tower "strong" will have an effect on the shape and size of the tower, but in the cases of "widest tower" or "tallest tower," the student does not have to make additional choices that will determine the height.

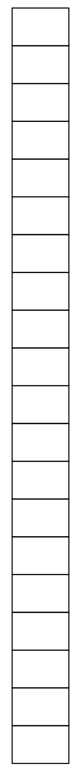

Figure 12. The tallest tower possible with 20 bricks

Elementary

In Appendix B, students are given a task to suggest items for the classroom with a $1,000 budget. As part of the problem-solving process, they must make some determinations about what makes items important or optimal to choose. They may have to do some preliminary work, such as deciding what items the classroom already has, what type of items would be useful in their classroom, and how much of any given item would be optimal. In this way, students are informally creating their own constraints as they create a mathematical model of a situation.

During the process of answering the question, students may choose to represent their solution using an equation, which may look something like the following, where for each item i, a_i is the number of those items purchased, and x_i is the cost of that item.

$$C = a_1x_1 + ... + a_nx_n$$

Students are maximizing this equation subject to the budget constraint, or $C \leq 1000$, and their informal constraints. At this developmental level, students are not expected to use formal language such as constraint, model, or maximization, but they do build those concepts as they informally approach them in the task.

Graduate

For the graduate-level task, students model a scenario of their choice, determine reasonable constraints, and present the optimal solution. This sample task examines feasibility in setting up a colony on the moon, and includes assumptions made in modeling, constraints, and methods of optimization used. Appendix B contains the bulk of the explanation and methods of each method of optimization for further information.

Applications of the Traveling Salesman project can help in the decision-making process for civil infrastructure planning. As NASA explains, "transportation plays a vital role in the operation of other technologies, such as manufacturing, construction, communications, health and safety" (Dunbar, 2009, p. 10). Therefore, minimizing transportation costs for fuel and time will have a positive impact on these other technologies. In a system with a minimum availability of personnel and resources, this kind of impact on costs in each of these areas will benefit the project as a whole.

The creation of a viable lunar colony is no longer in the realm of science-fiction. Scientists anticipate a land-based space colony within the next decade, and planning is well underway for the successful implementation of such a colony. NASA and SpaceX have been collaborating on a Mars base, and work has begun on the rockets needed to make the journey. Plans are in place to send two rockets with cargo and two with personnel in order to begin construction of the base by 2024 (Baidawi & Chang, 2017).

Landing on the moon is a complicated process, which include several constraints. In order to choose a viable landing site, students have identified a prior lunar landing site that is closest to the other main required location: the water ice deposits near the northern pole of the moon. Prior landing locations all occur on the side of the moon facing the Earth.

The locations for the colony are determined by two major required locations: the site for mining ice and the landing/launch site. Each node formed by a site requires about 15 miles of space for operations and future growth. Therefore, the tentative colony configuration with a minimum number of buildings can be seen in the left-most colony of Figure 13.

The second colony model has an additional building for a central personnel habitation node located at the midpoint of the landing and mining sites. The third model has two personnel habitation modules, one located at the intersection points of the far ends of the colony, seen in Figure 13 as the right-most colony model.

The sites are labeled as follows: $M =$ mining site, $L =$ landing site, $A =$ refinery, $B =$ Manufacturing, $C =$ research, and $D =$ personnel services (such as medical, meals, and educational facilites). In Colony Model 2, the point P_1 is the central personnel habitation module, and in Colony Model 3, the points P_2 and P_2' are the dual personnel habitation modules.

Given the required distances for M and L, and the preferred locations for sites $A-D$, it is possible to calculate the distances between each site. The lengths of each distance can be seen in Table 5 (See Appendix B).

Solution Method

For each colony option, the Traveling Salesman Problem (TSP) model will be used to simulate the distribution of materials beginning at the landing site, where supplies will come in to the colony, be distributed to every node, picked up from refinery and manufacturing, and be shipped out of the landing site again. Because the number of nodes in

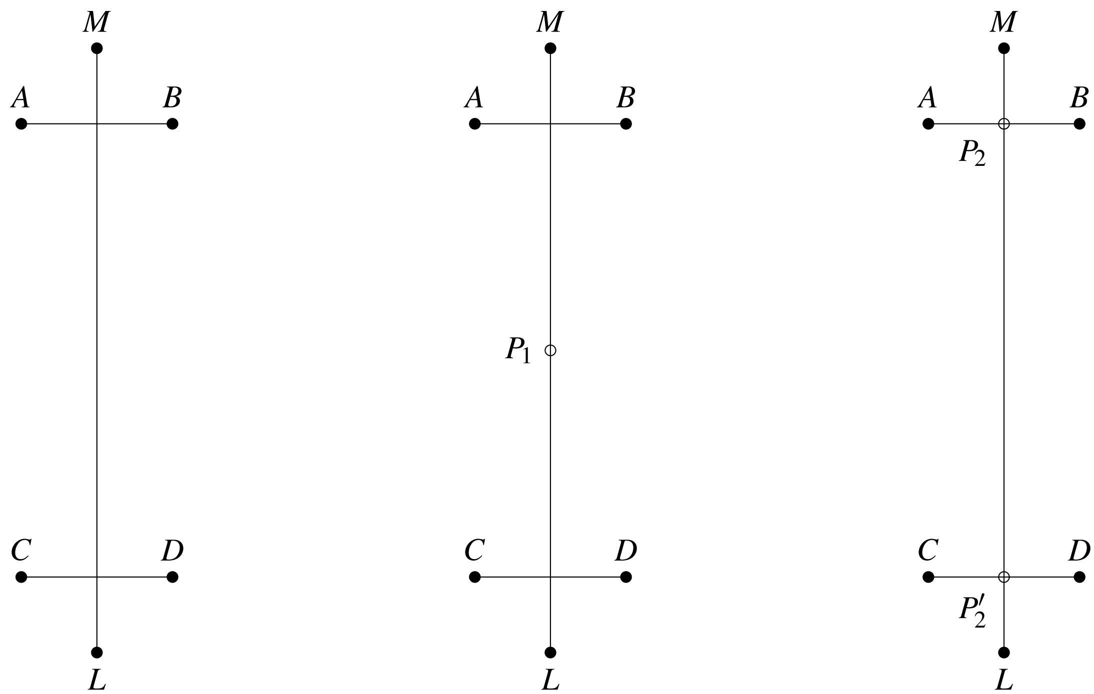

Figure 13. 3 configurations for colony based on personnel habitation locations

this problem increases with each option (and thus exponentially increases the maximum number of feasible solutions), a simple heuristic construction algorithm called the Nearest Neighbor Algorithm (NNA) will be employed to define the tours for materials distribution.

The NNA begins at any given node and moves to the nearest possible unvisited node. This pattern continues until all nodes have been visited and the algorithm is terminated. For Colony Option 1, the following cost matrix shows the distance between each site:

$$\text{Cost}_1 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ M \\ L \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & M & L \end{array} \\ \begin{pmatrix} - & 60 & 1431 & 1432.3 & 42.4 & 1462.3 \\ 60 & - & 1432.3 & 1431 & 42.4 & 1462.3 \\ 1431 & 1432.3 & - & 60 & 1462.3 & 42.4 \\ 1432.3 & 1431 & 60 & - & 1462.3 & 42.4 \\ 42.4 & 42.4 & 1462.4 & 1462.4 & - & 1491 \\ 1462.3 & 1462.3 & 42.4 & 42.4 & 1491 & - \end{pmatrix} \end{array}$$

Because of the added constraint to begin at the landing site, the algorithm begins at node L and obtains the resulting tour: $\mathbf{L}-\mathbf{C}-\mathbf{D}-\mathbf{B}-\mathbf{M}-\mathbf{A}-\mathbf{L}$. The total distance is 3080.5 miles.

Adding a central personnel habitation node for Colony Option 2 gives the second cost matrix:

$$\text{Cost}_2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ M \\ L \\ P_1 \end{array} \begin{array}{c} \begin{array}{ccccccc} A & B & C & D & M & L & P_1 \end{array} \\ \begin{pmatrix} - & 60 & 1431 & 1432.3 & 42.4 & 1462.3 & 716.13 \\ 60 & - & 1432.3 & 1431 & 42.4 & 1462.3 & 716.13 \\ 1431 & 1432.3 & - & 60 & 1462.3 & 42.4 & 716.13 \\ 1432.3 & 1431 & 60 & - & 1462.3 & 42.4 & 716.13 \\ 42.4 & 42.4 & 1462.4 & 1462.4 & - & 1491 & 745.5 \\ 1462.3 & 1462.3 & 42.4 & 42.4 & 1491 & - & 745.5 \\ 716.13 & 716.13 & 716.13 & 716.13 & 745.5 & 745.5 & - \end{pmatrix} \end{array}$$

The resulting tour for Colony Option 2 beginning at the landing site is $\mathbf{L}-\mathbf{C}-\mathbf{D}-\mathbf{P}_1-\mathbf{A}-\mathbf{M}-\mathbf{B}-\mathbf{L}$ with a total distance of 3081.76 miles.

Finally, Colony Option 3 is a model of a colony with dual personnel habitation modules. Using NNA, the tour for material distribution according to the cost matrix can be found below:

$$\text{Cost}_3 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ M \\ L \\ P_2 \\ P_2' \end{array} \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & M & L & P_2 & P_2' \end{array} \\ \begin{pmatrix} - & 60 & 1431 & 1432.3 & 42.4 & 1462.3 & 30 & 1431 \\ 60 & - & 1432.3 & 1431 & 42.4 & 1462.3 & 30 & 1431 \\ 1431 & 1432.3 & - & 60 & 1462.3 & 42.4 & 1431 & 30 \\ 1432.3 & 1431 & 60 & - & 1462.3 & 42.4 & 1431 & 30 \\ 42.4 & 42.4 & 1462.4 & 1462.4 & - & 1491 & 30 & 1461 \\ 1462.3 & 1462.3 & 42.4 & 42.4 & 1491 & - & 1461 & 30 \\ 30 & 30 & 1431 & 1431 & 30 & 1461 & - & 1431 \\ 1431 & 1431 & 30 & 30 & 1461 & 30 & 1431 & - \end{pmatrix} \end{array}$$

The resulting tour is $\mathbf{L} - \mathbf{P}_2' - \mathbf{C} - \mathbf{D} - \mathbf{B} - \mathbf{P}_2 - \mathbf{M} - \mathbf{A} - \mathbf{L}$ with a total distance of 3115.7 miles. The path of all three tours generated by the Nearest Neighbor Algorithm and their tour lengths are shown in Figure 14.

Solution

The final determination for an efficient colony model of transportation takes into consideration two particular types of optimization: a Traveling Salesman approach to material delivery of the colony and a piecewise approximation of colonist travel patterns based on population size. In the case of the material delivery model, the students consider that a delivery to the colony would land at the launch site and proceed cyclically to each node, picking up materials to be offloaded at one or more sites, and return to the launch site. Since there is only one 2.3-day window for launch per month (Compton, 2005), this trip would feasibly only take place once every month. As expected, the model with the fewest

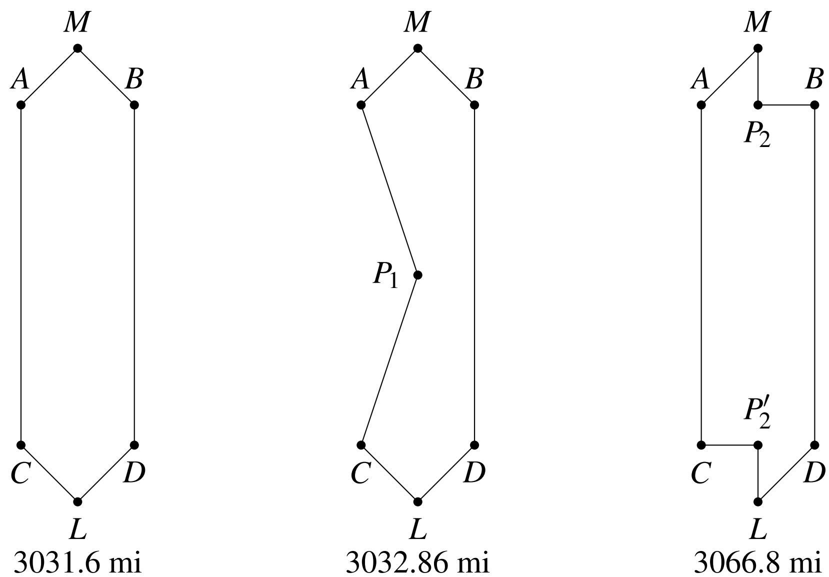

Figure 14. Final TSP tours for materials distribution to colony configurations

nodes, Colony Model 1, has the shortest optimized route at 3080.5 miles. Adding an additional node P_1 in Colony Model 2 increases the total distance by a little more than a mile at 3081.76 miles. Finally, Colony Model 3 has the largest increase of 35.2 miles for a total travel distance of 3115.7 miles when using nodes P_2 and P_2'.

For a moon base that only has workers stationed where they are needed, Colony Model 1 is the most efficient. It has the shortest delivery route, and it has no additional miles traveled by personnel since they live at their worksites. For a moon base that is intended as a viable colony and is expected to have population growth over time, personnel living arrangements and travel are a serious consideration. In a colony situation, it can be presumed that personnel would live in family arrangements. For modeling purposes, this is assumed to be two adults and (on average) two minor children. Since adult personnel would be required to work at the work modules, again assume that (the adult) half of the population travels to the six nodes listed as work sites, while the children do not travel. In the case of Colony Model 1, this implies that three-fourths of the population does not travel, but one-fourth of the population must travel evenly to the other nodes. In the case of

Colony Model 2, approximately half of the colony travels between all six work nodes, and in the case of Colony Model 3, half of the population travels to the work nodes near their habitation.

Assuming that workers remain at their worksites for the entire month and return to their habitation modules only once per month (which is, frankly, unlikely), the travel distances for the colonists can be considered in each of these scenarios. The minimum amount of travel monthly for colonists in Model 1 is over 32000 miles, Model 2 is over 58000 miles, and Model 3 is 2400 miles. As the population increases, the differences between these models will also increase.

Colony Model 3 has 29691.08 less miles of colonist travel than the next shortest model, but has an increase in delivery miles of 35.2. The miles added in the less efficient TSP model are more than outweighed by the distance colonists must travel to work sites. Therefore, for a planned-growth colony model, Model 3 is the most efficient model for transportation, while for a small working base, Model 1 is the most efficient model for transportation. Since the difference in miles is so great between the two models once family living arrangements take place, the following practices are recommended when beginning to build the base if even tentative plans for population growth exist: either choose Colony Model 3 to build, or add modules P_2 and P_2' before the population reaches 160.

CHAPTER V

DATA ANALYSIS

Meta-analysis Method

Data from cross-curricular studies compared the effect size of PBL on achievement. One meta-analysis technique, the Hedges-Olkin Fixed Effect method (Field, 2000; Hedges & Olkin, 1985), provides a way to pool the effect size of numerous studies. Because I am taking a weighted average effect size, I can consider the values from experiments that returned a non-significant effect. Meta-analysis provides a way to account for persistent results that may not have been significant in each individual study.

Using the reported sample size of the control and experimental groups, along with the reported effect size, I calculated a weighted average effect size and standard deviation for the group of studies, some of which were also included in other analyses (Gijbels, Dochy, den Bossche, & Segers, 2005) See Appendix A for the full list of studies and data used in this analysis.

For each study, I first calculated an unbiased effect size using this equation.

$$d_{\text{unbiased}_i} = \left(1 - \frac{3}{4-(N-2)-1}\right) \times d_i$$

See Table 4 in Appendix A for calculated values. In this equation, N is the total number of participants in the study, and d_i is the effect size for each individual study.

I then calculated the variance of the effect sizes, $\sigma^2_{d_i}$. These calculated variances are also reported in Table 4 (see Appendix A).

$$\sigma^2_{d_i} = \frac{n^e_i + n^c_i}{n^e_i \cdot n^c_i} + \frac{d^2_i}{2(n^e_i + n^c_i)}$$

Using the variances and unbiased effect sizes for each study, I calculated a weighted average effect size, d_+, for all of the studies.

$$d_+ = \frac{\sum_{i=1}^{k} \frac{d_i}{\sigma^2_{d_i}}}{\sum_{i=1}^{k} \frac{1}{\sigma^2_{d_i}}}$$

The weighted average effect size for the study of correlation of PBL and achievement is 0.1577. I then calculated an estimated standard deviation, $\hat{\sigma}_{d_+}$, using the estimated variances.

$$\hat{\sigma}_{d_+} = \sqrt{\sum_{i=1}^{k} \frac{1}{\sigma^2_{d_i}}}$$

The estimated standard deviation is 0.0167.

In the case of studies with a reported Pearson's r correlation coefficient, a mean difference d effect size can be calculated and pooled to get a weighted average effect size (Borenstein, Hedges, Higgins, & Rothstein, 2009), where d is the mean difference, and V_d is the corresponding variance.

$$d = \frac{2r}{\sqrt{1-r^2}}$$

$$V_d = \frac{4V_r}{(1-r^2)^3}$$

A point estimate for the variance of a study which gives only the Pearson's r correlation coefficient and sample size n_i can be calculated with this equation: (Hartung, Knapp, & Sinha, 2008).

$$\hat{\text{var}}(r_i) = \frac{(1-r_i^2)^2}{n_i - 1}$$

For comparisons given using an ANOVA, the η^2 statistic can be converted to Cohen's d and included in the weighted average effect sizes by finding f^2 and d (Cohen, 1988).

$$f^2 = \frac{\eta^2}{1-\eta^2}$$

$$d = 2f$$

Using these methods, studies that reported data differently were pooled to report effects on achievement (Anderson, Scott, & Hutlock, 1976; Block, 1972; Breland & Smith, 1975; Hong-Zheng & Guey-Fa, 2019; Jones, Gordon, & Schechtman, 1975). Table 2 shows effect size of various measures with achievement. For interest and achievement, the mean effect size is 0.7871 with a variance of 0.0003. For motivation and achievement, the mean effect size is 0.3747 with a variance of 0.0031, and for affect and achievement, the mean effect size is 0.3119 with a variance of 0.0118.

Table 2
Effect size for achievement and other measures

Study	*N*	Unbiased *d*	Variance
	Interest and Achievement		
Köller et al., 2001	602	0.4077	0.0015
	602	0.6513	0.0014
	602	0.8979	0.00112
Schiefele & Csikszentmihalyi, 1995	108	0.6707	0.0075
	45	0.4858	0.0202
	108	0.6707	0.0075
	106	0.4693	0.0085
	90	0.3218	0.0107

Continued on next page

Table 2 – *Continued from previous page*

Study	*N*	Unbiased *d*	Variance
	108	0.7179	0.0073
Motivation and Achievement			
Schiefele & Csikszentmihalyi, 1995	106	0.3012	0.0091
	90	0.1792	0.0111
	108	0.5792	0.0079
Affect and Achievement			
Malmivuori, 2006	723	0.3656	0.0013
	723	0.5385	0.0012
Self-efficacy and Achievement			
Malmivuori, 2006	723	0.5827	0.0012

Discussion

A rule of thumb for effect sizes is shown in Table 3 (Sawilowsky, 2009). This rule of thumb indicates that the effect size of interest and achievement is medium, the effect size of motivation and achievement is small, and the effect size of affect and achievement is also small. PBL, which has the largest body of research in this analysis, had the mean effect size of $d_+ = 0.1577$. This indicates that there is a small but persistent positive effect of PBL on achievement. It is worth pointing out, however, that the variances of each of these correlations is very low, and therefore this rule of thumb may be underestimating the overall effect.

Other effects on student learning may be of interest, but have less available research, and therefore a meta-analysis of their effect sizes is not feasible. All of these findings have been reported here as the unbiased d effect size for ease of comparison across studies. Some findings which had medium or larger effect sizes include enjoyment and self-efficacy, with $d = 0.8981$ (Malmivuori, 2006), motivation and interest, with $d = 0.8411$ (Schiefele & Csikszentmihalyi, 1995), PBL and motivation, with a very large effect size of $d = 1.6699$ (Rathunde & Csikszentmihalyi, 2005), PBL and motivation, with $d = 1.512$ (Guthrie, Klauda, & Ho, 2013), and motivation and self-efficacy, with $d = 1.007$ (Guthrie et al., 2013). Mastery-based learning also showed large effect sizes with affect ($d = 1.0056$), self-determination ($d = 1.1522$), and motivation ($d = 1.3844$) in further studies (Shih, 2008).

Table 3
Effect Size descriptions

Effect size	d
Very small	0.01
Small	0.2
Medium	0.5
Large	0.8
Very Large	1.2

Achievement is also positively correlated with mastery learning. Block and Burns (1976) conducted a meta-analysis of over 60 studies, and found that 59 of these showed a significant positive correlation between mastery learning and achievement, although actual values were not reported. In the same analysis, only three studies were reported to have a negative correlation between mastery learning and achievement.

Five issues driving motivation are likelihood of success, self-efficacy, perceived usefulness, goal orientation, and emotional connection. PBL addresses perceived usefulness, since the tasks are real-world based, and emotional connection, since student input and exploration are perceived as valuable to the work. Mastery learning addresses goal orientation and likelihood of success. Students have the opportunity to see concrete criteria for success and the steps needed to reach it. Self-efficacy is addressed with student self-assessment. Using similar criteria to mastery learning, students learn to accurately gauge their progress and develop more autonomy over their academic growth.

To optimize the learning experience, there must be structure, student competence, and autonomy (Patrick, Skinner, & Connell, 1993). These can be designed in the mathematics classroom with appropriately aligned tasks, mastery of goals, and student input and self-assessment. An optimal mathematical learning experience would include a well-developed and aligned problem based curriculum, clear goals and objectives to encourage mastery, and the opportunity for students to self-assess regularly.

CHAPTER VI

FURTHER STUDY AND LIMITATIONS

Further Study

Studies show that positive affect, problem-based approaches to learning, mastery learning, student self-assessment, and achievement all have strong positive correlations with each other. While many studies exist detailing the correlation between any two of these indicators of student learning, a further study would include the systematic incorporation of all of them.

I would like to propose a study across age groups using a highly-rated problem-based curriculum, in which the teachers are trained on using mastery as a gauge for success, and students regularly self-assess, as compared to a traditional classroom. Before and after the study, measures should be taken for engagement, affect, and achievement. I would predict that the study classroom would have higher outcomes in all areas compared to a traditional classroom.

I would be interested to also conduct a study on teacher affect, motivation, and burnout rate in the traditional and study classrooms.

Finally, a vertically-aligned study of problem-based curriculum throughout the mathematical education of a student would be instrumental in determining whether using problem-based mathematics tasks would be recommended as a whole from early learning through collegiate mathematics. This kind of longitudinal study could give important data about student affect, motivation, and achievement in mathematics and other STEM fields.

Limitations

One major limitation of this paper is lack of current data about affect and mathematics learning as it relates to problem-based tasks. There is evidence that problem-based tasks are correlated with achievement across studies, and somewhat in mathematics, but there is not sufficient data to conclusively state how the interactions work. Does changing to a problem-based curriculum increase affect in a student? Does this same student experience an increase in motivation only during this course, or throughout later courses as well? How frequent do problem-based tasks need to occur in the lifetime of the student to have a long-lasting or permanent effect on the student's achievement, transference skills, and motivation towards the subject? Is the effect increased when combined with other methods that are also shown to improve student retention, affect, and achievement? These questions are among those that can be answered with a longitudinal study using a problem-based curriculum and teaching tools such as mastery learning and student self-assessment.

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APPENDIX A

TABLES

Table 4

Correlations between Problem-Based Learning and Achievement

Study	N_e	N_c	Cohen's *d*	Unbiased *d*	Variance
Aaron et al., 1998	113	121	-0.44	-0.4386	0.0175
Albano et al., 1996	17	12	0.796	0.7474	0.1524
Antepohl & Herzig, 1997	110	110	0.603	0.6009	0.0190
Antepohl & Herzig, 1999	55	557	0.603	0.804	0.3735
Baca et al., 1990	37	41	-0.919	-0.9099	0.05683
Baroody et al., 2013	32	32	0.90	0.8891	0.0688
Baroody et al., 2013	32	32	0.56	0.5532	0.0650
Baroody et al., 2013	32	32	0.34	0.3359	0.0688
Baroody et al., 2013	32	32	0.50	0.4939	0.0645
Barrows & Tamblyn, 1976	10	10	1.409	1.3495	0.2496
Boshuizen et al., 1993	4	4	2.268	1.9722	0.8215
Chis et al., 2018	53	53	0.4163	0.4133	0.0386
Distlehorst & Robbs, 1998	47	154	0.18	0.1793	0.0279
	47	154	0.39	0.3885	0.0281
	47	154	0.5	0.4981	0.0284
	47	154	0.3	0.2989	0.2839
	47	154	0.33	0.3288	0.0280
	47	154	0.14	0.1395	0.0278
Doucet et al., 1998	34	29	0.434	0.4286	0.0654
	21	26	1.293	1.2713	0.1039
Drake & Long, 2009	15	14	0.72	0.6998	0.1470
Finch, 1999	21	26	1.904	1.8721	0.1246

Continued on next page

Table 4 – *Continued from previous page*

Study	N_e	N_c	Cohen's *d*	Unbiased *d*	Variance
	72	501	-0.404	-0.439	0.0159
Goodman et al., 1991	72	501	-0.242	-0.242	0.0159
	12	12	0	0	0.1667
	15	13	0.769	-.7466	-.1541
	15	13	-.667	-0.6475	0.1515
	36	297	-0.133	-0.1327	0.0312
	12	12	-0.071	-0.0685	0.1668
Hmelo et al., 1997	20	20	0.883	0.8655	0.1097
	20	20	0.578	0.566 5	0.1042
Hmelo, 1998	39	37	0.521	0.5157	0.0545
	39	37	0.762	0.7543	0.0565
	39	37	0.547	0.5414	0.0546
	39	37	1.241	1.2284	0.0628
Lewis & Tamblyn, 1987	22	20	0.24	0.2355	0.0961
	22	20	.234	0.2296	0.0961
Martenson et al., 1985	1651	818	-0.15	-0.1500	0.0018
	1651	818	0.15	0.1500	0.0018
Mennin et al., 1993	144	447	0.056	0.0459	0.0092
	103	313	0.307	0.3064	0.0130
Moore et al., 1993	60	61	0.455	0.4521	0.0339
	60	61	-0.138	-0.1371	0.0331
	60	61	0.323	0.3210	0.0335
	60	61	-0.257	-0.2554	0.0333

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Table 4 – *Continued from previous page*

Study	N_e	N_c	Cohen's d	Unbiased d	Variance
	60	61	0.029	0.0288	0.0331
	60	61	-0.037	-0.0368	0.0331
	60	61	-0.159	-0.158	0.0332
Richards et al., 1996	88	364	0.05	0.0499	0.0141
	88	364	0.426	0.4253	0.0143
	88	364	0.425	0.4243	0.0143
	88	364	0.462	0.4612	0.0143
	88	364	0.073	0.0729	0.0141
	88	364	0.39	0.3894	0.0143
Santos-Gomez et al., 1990	41	78	0.525	0.5216	0.0384
	39	71	0.257	0.2552	0.0400
	43	70	0.525	0.5214	0.0388
Saunders et al., 1990	45	243	-0.716	-0.7141	0.0272
	47	242	-0.476	-0.4748	0.0258
	44	243	1.017	1.0143	0.0286
Schmidt et al., 1996	30	582	0.31	0.3096	0.0351
Schuwirth et al., 1999	30	32	0.06	0.0592	0.0646
	30	30	0.25	0.2468	0.0672
	29	30	0.238	0.2349	0.0683
	27	30	0.732	0.7220	0.0751
	32	25	1.254	1.2368	0.0850
Son & VanSickle, 2000	72	68	0.381	0.3789	0.0291
	72	80	0.384	0.3821	0.0269

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Table 4 – *Continued from previous page*

Study	N_e	N_c	Cohen's *d*	Unbiased *d*	Variance
Tans et al., 1986	6	5	2.171	1.9849	0.5809
	74	45	-2.583	-2.5664	0.0638
Verhoeven et al., 1998	135	122	-0.385	-0.3839	0.0159
	190	124	0.203	0.20257	0.0134
	145	104	0.211	0.2104	0.0166
	135	87	0.288	0.2870	0.0191
	188	151	0.193	0.1926	0.0120
	144	140	-0.037	-0.0369	0.0141
Wirkala & Kuhn, 2011	76	76	1.2841	1.2778	0.0317

APPENDIX B

TASKS

Elementary Tasks: Rotation

From Illustrative Mathematics, Grade 1 Geometry

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Materials

- Paper cut outs of rectangles, circles, and squares
- Blank paper

Actions

Part One:

Give each pair of students a square and ask, "How can you share the square equally so that you and your partner get the same size piece?"

Ask students to fold the paper to show how they could get two equal parts.

Call on student volunteers to share, asking the class as each example is displayed, "Is the paper shared equally? Will each person get the same size piece? How do you know?"

Create a chart, showing some of the ways students folded the square to make two equal parts. Students may fold vertically, horizontally, or diagonally.

Tell students, "There are two equal parts." Ask students, "What can we call each part of the rectangle?" Elicit student thinking, building the understanding that each piece is one of two equal parts, or half of the rectangle. The standard calls for students to "describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of." Helping students develop this language is critical. If students don't come up with this language, it's important to introduce it. After introducing the terms "half" and "half of" in this case, adding labels to the chart will help anchor the language.

You may repeat this with other shapes (circle, rectangle) or increase the number of students sharing the paper shape from 2 to 4. If you repeat with a rectangle some students may use scissors as scaffolding to cut the rectangle instead of folding when they divide the rectangle diagonally.

Part Two:

Pose the problem to students: “If you and three friends want to share a cake so that you each get the same amount, how much can each person have?” Ask questions to ensure students understand the context and problem being posed. Questions might include:

What is the story about? How many people are sharing the cake? What does it mean for each person to get the same amount? Provide students with tools such as blank paper and/or paper shapes to solve the problem. The cake context was selected to allow students to explore with multiple shapes as cakes may be round, square, or rectangular in shape.

As students solve the problem, monitor their progress looking for students that partition a shape into four equal shares.

Conduct a share out, showing several different student solutions that show a shape partitioned into four equal parts. It may be helpful to consider the features of each solution being shared by posing questions that get at student thinking and the essential mathematical ideas.

The teacher might ask questions such as:

How many people are sharing the cake?

What does the picture represent? What is the circle? What is the square? What is the rectangle?

How is __’s picture similar or different than __’s?

How much cake does each person get?

Some people represented the cake with rectangles and others with squares or circles, did that change the amount of cake each person gets? Why or why not?

What can we call these parts? How can we label them?

Why do the parts need to be equal?

IM Commentary

The purpose of this task is for students to understand how to partition shapes into equal pieces. This task starts students with concrete representations of the shapes that they can fold and cut, so that later they will understand more abstract representations like diagrams and symbols. Part one provides students with opportunities to manipulate paper shapes, folding them to create equal parts. Students start with a square so that vertical, horizontal, and diagonal folds all match the two sides exactly, this makes it easy to see what is meant by "equal parts" in this case. Rectangles pose a greater challenge, because the diagonal fold does not match the two halves up exactly. The teacher should lead a discussion about how we know the two halves are equal by cutting them apart and showing they match up exactly (or providing students with scissors and allowing them to do this for themselves). The idea of "equal parts" with shape is actually very subtle; in grade 1, students only look at congruent figures as equal. Later they can talk about equal area even when shapes aren't congruent; students begin to tackle these issues in Grade 3 and beyond.

Through a class discussion, students will understand that shapes can be partitioned in many different ways and that there are conventions for naming those parts. The task intentionally begins with partitioning a shape into two equal parts to build on students intuitive knowledge and experience working with halves. Students may be familiar with the term half but think of it as two parts rather without understanding that the parts must be equal in some sense. Developing student language is a critical component of this task, the standard calls for students to "describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of."

Part one can be extended to provide students with opportunities to partition additional shapes and increase the partitioning to create four equal shares. Extending the activity to include partitioning four equal shares can create opportunities for students to understand that decomposing into more equal shares results in smaller parts.

Part two provides students with a context, sharing cake, to further build on the ideas above. Students will see there are many ways to represent and partition shapes. However, the parts must be equal. Students will reason abstractly and quantitatively as they create equal shares in and out of context.

Solution

Part One: Students may fold the rectangle vertically, horizontally, or diagonally to show two equal parts.

Part Two: Students may draw a circle (or other shape) to represent the cake and then partition the shape into four equal parts. It's important for students to understand that the parts must be equal so that everyone gets a fair share. Students may label each part one of four equal parts, fourths, or one out of four equal parts.

Secondary Tasks: Rotation

From Illustrative Mathematics and Kendall Hunt, High School Geometry

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Launch:

Briefly demonstrate how to create an angle of a given measure with a protractor and ask students to use their protractors along with you.

Student Task:

1. Draw a segment. Label the endpoints A and B .

(a) Rotate segment AB clockwise around center B by 90 degrees. Label the new endpoint A' .

(b) Connect A to A' and lightly shade in the resulting triangle.

(c) What kind of triangle did you draw? What other properties do you notice in the figure? Explain your reasoning.

2. Draw a segment. Label the endpoints C and D.

   (a) Rotate segment CD counterclockwise around center D by 30 degrees. Label the new endpoint C'.

   (b) Rotate segment $C'D$ counterclockwise around center D by 30 degrees. Label the new endpoint C''.

   (c) Connect C to C'' and lightly shade in the resulting triangle.

   (d) What kind of triangle did you draw? What other properties do you notice in the figure? Explain your reasoning.

Solution:

1. It is a right isosceles triangle. It's a right triangle because the angle of rotation was 90 degrees. It's isosceles because a rotation is a rigid transformation so segment AB is congruent to segment $A'B$. The base angles appear congruent.

2. The result should be a segment labeled CD.

   (a) The result should be a new line segment DC' such that angle CDC' measures 30 degrees.

   (b) The result should be a new line segment DC'' such that angle $C'DC''$ measures 30 degrees.

(c) The result should be an equilateral triangle DCC''.

(d) It is an equilateral triangle because two sides are equal and the angle is 60 degrees. The segment $C'D$ is an angle bisector because of the construction. The segment $C'D$ is a perpendicular bisector (students know a variety of ways to verify this).

Synthesis:

Display an isosceles triangle ABC with base BC.

Here are some key conjectures and observations students should come away from the discussion with:

- Rotations preserve the distance to the center of rotation.
- The angle bisector of angle A is also the perpendicular bisector of the base (BC) of an isosceles triangle.
- The two base angles (angle B and angle C) of an isosceles triangle are congruent.
- An isosceles triangle where angle A is 60 degrees is also equilateral.

Elementary Tasks: Optimization

From Illustrative Mathematics, Grade 3

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Task Your teacher was just awarded $1,000 to spend on materials for your classroom. She asked all 20 of her students in the class to help her decide how to spend the money. Think about which supplies will benefit the class the most.

Write down the different items and how many of each you would choose. Find the total for each category: supplies, books and maps, puzzles and games, and special items.

Supplies	Cost
A box of 20 markers	$5
A box of 100 crayons	$8
A box of 60 pencils	$5
A box of 5,000 pieces of printer paper	$40
A package of 10 pads of lined paper	$15
A box of 50 pieces of construction paper	$32
Books and maps	**Cost**
A set of 20 books about science	$250
A set of books about the 50 states	$400
A story book (there are 80 to choose from)	$8
Maps	$45
Puzzles and games	**Cost**
Puzzles (there are 30 to choose from)	$12
Board games (there are 40 to choose from)	$15
Interactive computer games (math and reading)	$75
Special Items	**Cost**
A bean bag chair for the reading corner	$65
A class pet	$150
Three month's supply of food for a class pet	$55
A field trip to the zoo	$350

Create a bar graph to represent how you would spend the money. Scale the vertical axis by $100. Write all of the labels. What was the total cost of all your choices? Did you have any money left over? If so, how much? Compare your choices with a partner. How much more or less did you choose to spend on each category than your partner? How much more or less did you choose to spend in total than your partner?

IM Commentary The purpose of this task is for students to "Solve problems involving the four operations" (3.OA.A) and "Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories" (3.MD.3). Additionally, students will engage in MP3, Model with mathematics. In this task students are asked to decide how to spend $1,000 on supplies and materials for their classroom; students will have to make choices and be careful not to exceed the budget. Students are asked to decide which supplies will benefit the class the most and will compare their choices with other students' choices.

Because the budget does not allow students to buy one of everything, this task provides an opportunity for teachers to discuss costs and benefits. A benefit is something that satisfies your wants. A cost is what you give up when you decide to do something.

In third grade students are asked to fluently add and subtract within 1,000 (3.NBT.3) which is why the total budget is \$1,000. Students are also multiplying and dividing within 100 (3.OA.7), so they might choose, for example, to buy 20 boxes of markers at \$5 each so that there is a box of markers for every student in the class. It is possible that students will choose to purchase a number of one of the items where the total is greater than \$100; while students are not expected to be fluent in multiplication above 100, they should be able to use their multiplication strategies to find such products. This task provides students with a natural opportunity to use addition, subtraction, and multiplication, and they might also use division depending on how they approach the task; thus it is well aligned to 3.OA.8.

Bar graphs make it easy for students to compare their allocations. If all of the students in the class include all categories on their graphs (whether they allotted any spending to them or not), list the categories in the same order that they are listed in the data table, and use the same colors for each category on a final draft, the teacher can put all of the final graphs up for display and the class can see whether there is a general consensus for how to spend the \$1000 or not.

As an extension, the teacher might consider asking students to represent their total purchases with an equation; for example, if a student chooses 15 boxes of markers, 3 boxes of crayons, and 2 beanbag chairs, she could write:

$$15 \times 5 + 3 \times 8 + 2 \times 65 = 75 + 24 + 65 + 65 = 229$$

This task is part of a set collaboratively developed with Money as You Learn, an initiative of the President's Advisory Council on Financial Capability. Integrating essential financial literacy concepts into the teaching of the Common Core State Standards can strengthen teaching of the Common Core and expose students to knowledge and skills they need to become financially capable young adults. A mapping of essential personal finance concepts and skills against the Common Core State Standards as well as additional tasks and texts will be available at http://www.moneyasyoulearn.org.

Solution Solutions will vary. Here is one possible set of choices.

- 8 boxes of markers will cost $8 \times 5 = 4 \times 2 \times 5 = 4 \times 10 = 40$ dollars.
- 4 boxes of crayons will cost $4 \times 8 = 4 \times 4 \times 2 = 16 \times 2 = 10 \times 2 + 6 \times 2 = 20 + 12 = 32$ dollars.
- 2 boxes of pencils will cost $2 \times 5 = 10$ dollars.
- 1 box of printer paper costs 40 dollars.
- 2 packages of lined paper cost $2 \times 15 = 2 \times 10 + 2 \times 5 = 20 + 10 = 30$ dollars.
- 3 boxes of construction paper cost $3 \times 32 = 3 \times 30 + 3 \times 2 = 90 + 6 = 96$ dollars.
- The total for the supplies is 40+32+10+40+30+96=248 dollars.
- 12 books cost $12 \times 8 = 10 \times 8 + 2 \times 8 = 80 + 16 = 96$ dollars.
- The total cost for the books and maps is 250+96+45=391 dollars.
- The total cost for the puzzles and games is $10 \times 12 + 6 \times 15 = 120 + 3 \times 30 = 120 + 90 = 210$ dollars.

The total for the special items is 130 dollars.

The total from all the purchases would be 248+391+210+130=979. So these purchases would total $979 and $21 would be left over. Comparisons will vary.

Graduate Tasks: Optimization

Exerpts from an unpublished paper, presented at Texas Woman's University in MATH 5863 (Elizondo & Skousen, 2017)

Optimizing the Colony

Being able to mine ice deposits on the moon is important for a functional, long-term base. "A NASA radar instrument ... found evidence of at least 600 million metric tons of water ice spread out on the bottom of craters at the lunar north pole. It is yet another supply of lunar water ice, a vital resource that could be mined to produce oxygen or rocket fuel to support a future moon base" (Malik, 2010). The lunar ice locations are located in permanently shadowed craters near the poles, the largest of which is found at $81.4°N, 22°E$ (Crusan, 2010). These two locations will represent the furthest locations of our colony modules. We will also need four other main sites: a refinery for the processing of lunar ice, a manufacturing facility for colony needs, a personnel services center (for food, supplies, medical, and other needs), and a research lab. We will use the TSP to compare the efficiency of three types of habitation models for the colony: having individual personnel habitations located in the module they work, having one large centralized habitation module for personnel, and having two medium habitation modules near the farther locations.

Solving the Traveling Salesman Problem

Solution methods for the TSP may be divided into a few overarching categories including exact solutions and heuristic algorithms. Exact solution methods will always produce an optimal solution. The well-known Branch-and-Bound method falls into this category as it is "an organized way to make an exhaustive search for the best solution in a specified set"(Applegate, 2007). There do exist, however, some instances in which obtaining optimal solutions through exact algorithms are not viable. For every n-city problem, there exists a maximum of $(n-1)!$ feasible solutions. Therefore, there has been the need to develop many heuristics algorithms which produce good, feasible solutions that may not always be optimal.

Segment	Distance (miles)
$\overline{ML}$	1491
$\overline{MA}$	42.4
$\overline{MB}$	42.4
$\overline{LC}$	42.4
$\overline{LD}$	42.4
$\overline{AC}$	1431
$\overline{BD}$	1431
$\overline{MC}$	1462.3
$\overline{MD}$	1462.3
$\overline{AL}$	1462.3
$\overline{BL}$	1462.3
$\overline{AB}$	60
$\overline{CD}$	60

Segment	Distance (miles)
$\overline{MP_1}$	745.5
$\overline{LP_1}$	745.5
$\overline{AP_1}$	716.1
$\overline{BP_1}$	716.1
$\overline{CP_1}$	716.1
$\overline{DP_1}$	716.1
$\overline{MP_2}$	30
$\overline{LP_2'}$	30
$\overline{P_2P_2'}$	1431
$\overline{AP_2'}$	1431.3
$\overline{BP_2'}$	1431.3
$\overline{CP_2}$	1431.3
$\overline{DP_2}$	1431.3

Table 5: Distances between colony location points

Colony Options

We observe several intersecting edges in the tours generated by the NNA and wish to improve the tours using the 2-opt local search improvement algorithm. For Colony Option 1, edges **A-L** and **C-D** intersect, so we swap them to create two new edges: **A-C** and **L-D**. Calculating the new length of the tour we obtain a shorter distance of 3031.6 miles. This 2-opt swap is seen in Figure 16, with the red dashed lines representing the edges to be swapped and the blue lines representing the newly connected edges. The resulting tour is $\mathbf{L}-\mathbf{D}-\mathbf{B}-\mathbf{M}-\mathbf{A}-\mathbf{C}-\mathbf{L}$.

We repeat this procedure for Colony Option 2, observing first that edges **D-P**$_1$ and **B-L** intersect. We swap them to create two new edges: **D-B** and **P**$_1$**-L**. Calculating the length of the new tour we obtain a shorter distance of 3079.83 miles, thus we keep the swap.

Although the first iteration of the 2-opt move improved the tour by 1.93 miles, Figure 17 shows there still exist intersecting edges in the tour. We continue with a second

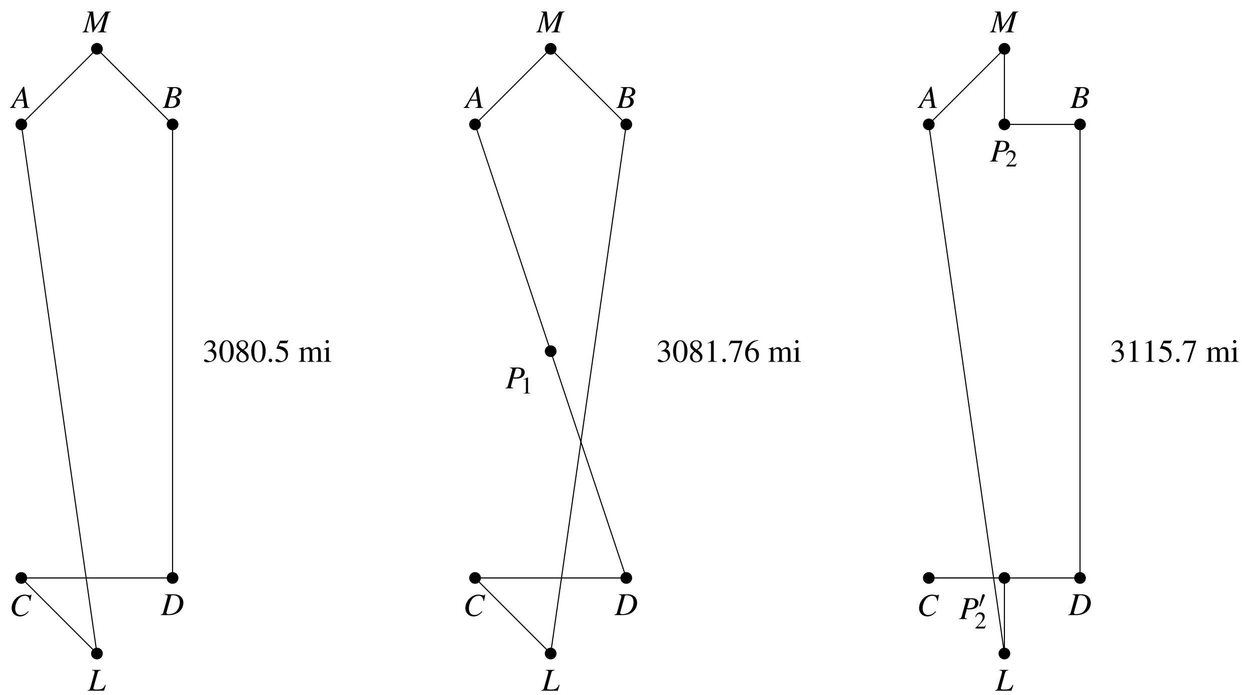

Figure 15. NNA generated tours for colony configurations

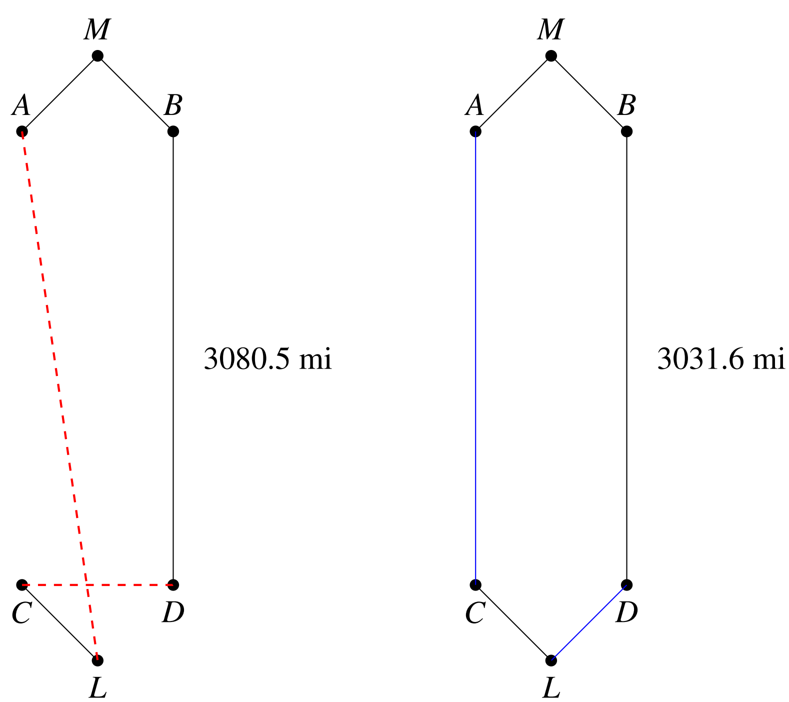

Figure 16. 2-opt local search for Colony Option 1

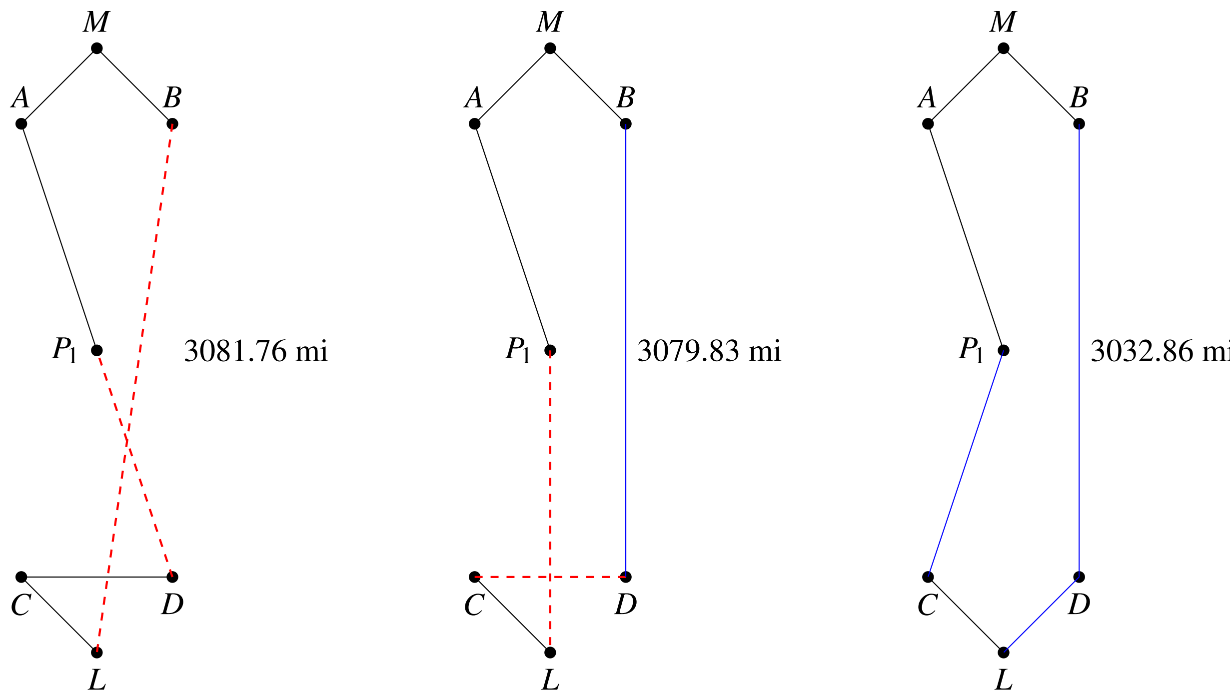

Figure 17. 2 iterations of 2-opt local search for Colony Option 2

iteration of the algorithm, then, by disconnecting the intersecting edges **P**$_1$**-L** and **C-B** and connecting **L-D** and **P**$_1$**-C**. The length of the new tour, **L** – **D** – **B** – **M** – **A** – **P**$_1$ – **C** – **L**, is 3032.86 miles and there are no remaining intersecting edges.

Finally, we run the 2-opt local search algorithm for Colony Option 3 by disconnecting the intersecting edges **C-D** and **A-L** and connecting **L-D** and **A-C**. The new tour, **L** – **D** – **B** – **P**$_2$ – **M** – **A** – **C** – **P'**$_2$ – **L**, has a length of 3066.8 miles, a 48.9 mile tour improvement. Figure 18 shows the 2-opt local search algorithm applied to Colony Option 3.

Thus, by first employing the Nearest Neighbor construction algorithm and then improving the tours using a 2-opt local search, we obtain Traveling Salesman tours for materials distribution to our three proposed colony configurations that are either optimal or very close to optimal (Figure 14)

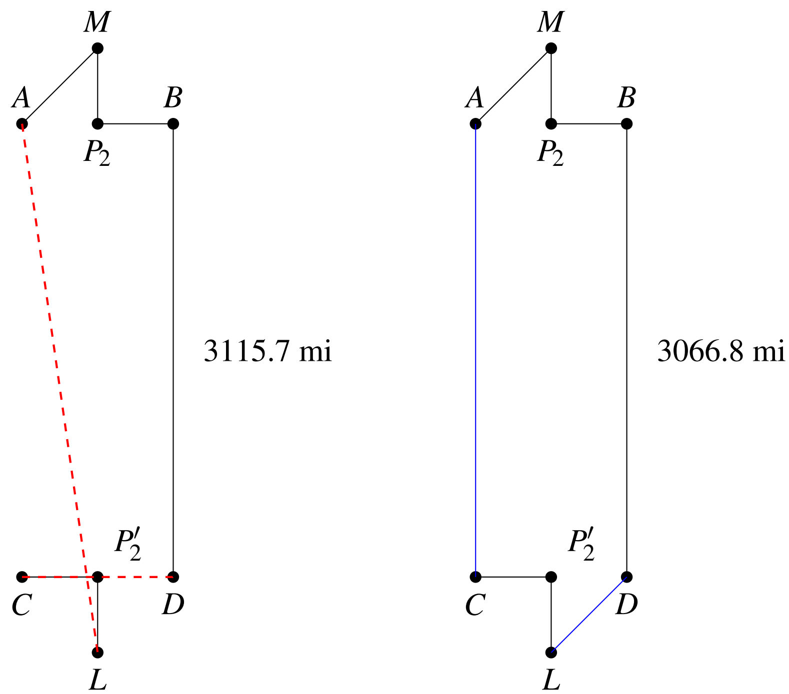

Figure 18. 2-opt local search for Colony Option 3

Optimizing for colonists

Here we take into consideration that the previous cyclic tour is an optimization for a transportation cycle that covers each node once. While this is an important tour for material distribution for the colony, it is not the only type of transportation that will occur. Specifically, colonists will need to travel to from their stationed work places and the locations where they sleep. Therefore, we will consider the three colony models again. Colony 1 has a living site at each node built in, so that any colonists are living at their work site. This requires the least amount of travel between living space and work station, since it is trivial, or that is to say, there is no additional distance. We consider that a minimum of three workers will be at each node in order to cover 3 8-hour shifts at each station, so the minimum number of colonists will be assigned as $n = 18$. Since a viable colony requires at least 160 colonists, (Carrington, 2002) population dynamics begin to change when $n > 160$. Assuming that until the population reaches 160, only adult workers are present, while for populations over 160, the colonists are arranged in family units, we will consider a simplified linear model of colony population that can be described by the following piecewise function:

$$f_0(n) = \begin{cases} 0, & 18 \le n \le 159 \\ \frac{42.5n}{13} + \frac{60n}{26} + \frac{1462.3n}{13} + \frac{1431n}{13} + \frac{1491n}{52}, & n \le 160 \end{cases}$$

where minor dependents do not travel to work sites and are on average half the population for $n \ge 160$. Additionally, half of the remaining adults would not travel, while the other half would travel evenly to the other 5 nodes.

Colony 2 has one central living site, P_1, where all the colonists would reside.

$$f_1(n) = \begin{cases} \frac{4}{6}n(716.13) + \frac{2}{6}n(745.5), & 18 \le n \le 159 \\ \frac{4}{12}n(716.13) + \frac{2}{12}n(745.5), & n \le 160 \end{cases}$$

For $n \ge 160$ minors do not travel, while the adults travel evenly between all 6 non-habitation nodes.

Similarly, we can construct a function describing travel for colonists with Colony model 3, with habitation modules at P_2 and P_2'.

$$f_2(n) = \begin{cases} 30n & ,18 \le n \le 159 \\ \frac{1}{2}n(30) & ,n \le 160 \end{cases}$$

In this scenario, for $n \ge 160$, the minors remain at the habitation module, while the adults at each node travel between the 3 work modules at their respective ends, each of which is 30 miles away.

As we can see from the functions described, for $18 \leq n \leq 159$, colony option 1 with no additional habitation modules has the least amount of colonist travel.

Since the goal of this project is to eventually create a sustainable colony model, it is important to consider the population at $n \geq 160$. We will calculate the infimum of each function for $n \geq 160$, which will give us the greatest lower bound of travel distance for colonists.

In the case of Colony 1, $f_0(160) = 32091.08$. For the model of Colony 2, $f_1(160) = 58073.6$. The model of Colony 3 yields $f_2(160) = 2400$.