# OPTIMAL CONTROL THEORY APPLIED TO A COOPERATIVE GAME BETWEEN A MANUFACTURER AND A RETAILER

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To the Dean of the Graduate School:

I am submitting herewith a thesis written by Cynthia Ann Hitt entitled "Optimal Control Theory Applied to a Cooperative Game between a Manufacturer and a Retailer." I have examined this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science with a major in Mathematics.

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ABSTRACT

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OPTIMAL CONTROL THEORY APPLIED TO A COOPERATIVE GAME BETWEEN A MANUFACTURER AND A RETAILER

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In this work, we consider a process of production, storage and sales of a perishable consumer good as a cooperative game between a manufacturer and a retailer. Both parties want to choose such a strategy (optimal production plan, optimal reselling plan, etc.) that will maximize their cumulative profits. The problem is reduced to the maximizing of the corresponding functionals of profit and solved with the use of the Optimal Control Theory. Computer programs are written in MAPLE in order to demonstrate and confirm our analytical results.

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# CHAPTER I

# INTRODUCTION

Global economic competition combined with advancing manufacturing technologies has created a need for dynamical models, simulating complex economic conditions. With the increasingly available mathematical software and computer memory capability the models are not just theoretical but practical. Most of the economic models can be created with differential equations. After describing the microeconomic situation with differential equations, the system will have many different solutions, depending on the various input parameters. Optimally we want to control the system, to make it applicable to our specific situation and result in practical, usable solutions (outputs).

Economic problems can be addressed with the use of dynamic optimization modeling. Optimal (or best choice) use of resources, production rate, at a point of time over time, involves both static and dynamic optimization. The choice between less production now and more production later can be classified as a dynamic optimization problem. If the optimal static conditions are determined we might automatically want to apply those to a dynamic situation, believing that it would be optimal for that case as well. This is not necessarily the case.

Different statically optimal models trace out different paths, not all of which are dynamically optimal. Pontryagin's Maximum Principal, which is the major principal of Optimal Control Theory, is an important tool of dynamic optimization. Optimal Control involves choosing among all the" admissible control variables $u(t)$ , the one which will bring a dynamic system from some initial state $x(t_0)$ at time $t_0$ to some terminal state $x(T)$, at some terminal time $T$ , in such a way as to impart a maximum or minimum to a certain objective functional which is also called a performance index."**[1]**.

The state of a dynamic system is a collection of numbers $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ which, once specified at time $t = t_0$, are determined for all times $t \geq t_0$ by the choice of the control vector $u(t) = (u_1(t), \ldots, u_r(t))$. The numbers $x_i(t)\ (1 \leq i \leq n)$ are coordinates. Similarly, $u(t)$ is an **r**-control vector. **[2]**

The state at time $t$ of a system is represented by a system of differential equations, which is known as the dynamic system. For example:

$$\dot{x}(t) = f[x(t), u(t), t]$$

The control variables for the dynamic systems are those which obey the various restrictions imposed by the physical conditions of the problem. If $u(t)$ is a function of time only, we have an open-loop control. An example of an open loop control is the setting of a washing machine or dishwasher in which each cycle is of a given time. If $u(t)$ is a function of some state variables, for example: $u(t) = u[x(t), t]$, it is a closed loop control. A thermostat control, which is a function of the room temperature, is a closed loop control. **[2]**

Control variables must be chosen in order to maximize or minimize some functional:

$$J = \int_{t_0}^{T} f_0(x, u, t)dt \rightarrow \max$$

In this work, we will be using game theory and optimal control theory in order to develop and investigate a realistic and practical model of the process of production, storage and sales of a perishable consumer good. Since optimal control theory has as its objective the maximization of profit or the minimization of the cost of an economic or physical process, we can summarize our work as follows:

1. Write a mathematical representation of the process to be controlled.
2. Define and state the physical constraints of the model.
3. Specify what performance criterion will be optimal.
4. Solve the closed loop control problem analytically and numerically.
5. Write computer programs in order to demonstrate results of the investigation.

## CHAPTER II

## REVIEW OF THE LITERATURE

Optimal control theory is a recent development for solving various problems in engineering and economics. Calculus of variation was useful for optimization, but when a bounded control was added, it could not be applied. In 1959 Pontryagin published a work outlining the theory later referred to as the Maximum Principal; this principal became the basis of the optimal control theory. Optimal control theory is being used in many applied mathematics and economic areas. **[3]**

Game theory foundations were first developed by John von Neumann, who in 1928 proved the basic MiniMax theorem, and with the publication in 1944 of the *Theory of Games and Economic Behavior*, the field was established. John von Neumann, a mathematician, collaborated with economist Oskar Morgenstern to write *The Theory of Games and Economic Behavior*. In the book they stated:"We hope, however, to obtain a real understanding of the problem of exchange by studying it from an altogether different angle; this is, from the perspective of a 'game of strategy'." Morton D. Davis in his book **[4]**, states:"the uniqueness of game theory problems are that there are others present who are making decisions in accordance with their own wishes, and they must be taken into account. While you (as one player) are trying to figure out what they are doing, they will be trying to figure out what you are doing". In a *Fortune* magazine article John McDonald (1970), writing about executive decision making, noted that game theory is 'uniquely qualified to make sense of the forces at work' and described how it could be 'related. .to the strategies of some actual corporations caught up in conglomerate warfare'. He saw airline competition, product diversification, and conglomerate absorption as fertile areas in which to use game theory. In the business world it is also used to derive optimal pricing and competitive bidding strategies and to make investment decisions. **[1, 4]** Cooperative game theory developed with the player's results being different from the traditional goal of many common games. With traditional games there is a winner and a loser, or a person who does not gain as much as the winner. In some games the winner's success is based on a direct ratio with the loss of the other player. In cooperative games with two or more players, the players do not compete against each other, but work together to a stated goal. An example of a cooperative game in a recreational setting is called 'stand-up'. A group of players sitting down in a circle, facing outward, with their arms linked, attempt to stand up. The players will each have to cooperate and help each other to achieve their goal. An example of a cooperative game in a business environment would be the success of a failing business. The financial recovery of Chrysler Motor Company from bankruptcy with Lee Iacocca as the chairman in the 1970's is a good example of every team player working for the success of the company. The players in a cooperative game, as the name suggests, cooperate with each other for the success of both or all parties. The players working together strive to a common goal that benefits all the parties involved. **[4]**

Optimal control theory is finding a chosen set of controls, so as to maximize or minimize an objective that has been chosen. Ponder the example of a fish stock from the book of Robert Shone **[3]** :"Consider a fish stock which has some natural rate of growth and which is harvested. Too much harvesting could endanger the survival of the fish, too little and profits are forgone. Of course, harvesting takes place over time. The obvious question is: 'what is the best harvesting rate, i.e., what is the optimal harvesting?' The answer to this question requires an *optimal path or trajectory* to be identified. 'Best' itself requires us to specify a criterion by which to choose between alternate paths. Some policy implies there is a means to influence (control) the situation." There are a number of ways to solve such a control problem: calculus of variations, dynamic programming and optimal control theory are the most common.

In this paper we will use optimal control theory and the Maximum Principle developed by Lev Pontryagin **[1]** to investigate a microeconomic control model. For instance, the model of production, storage and sales of a consumer good, proposed and completely investigated by Dr. Ellina Grigorieva **[5, 6]**, is very complex and interesting and can be chosen as our basic model:

$$\begin{cases} \dot{x}_1(t) = -n_p(Y - x_2(t))x_1(t) + u(t), \quad t \in [0,T] \\ \dot{x}_2(t) = n_p(Y - x_2(t))x_1(t) - k_1 x_2(t) \\ \dot{x}_3(t) = p n_p(Y - x_2(t))x_1(t) - k_2 x_1(t) - u(t) \\ x_1(0) = x_1^0 \geq 0, \; x_2(0) = x_2^0 \geq 0, \; x_3(0) = x_3^0 \geq 0 \end{cases} \tag{2. 1}$$

where $x_1, x_2,$ and $x_3$ are variables and $T, Y, k_1, k_2, n_p$ and $p$ are constant parameters.

They mean the following:

$x_1(t)$ represents the amount of a consumer good on the market,

$x_2(t)$ represents the quantity of the good unused in consumers' homes,

$x_3(t)$ represents the profit,

**u(t)** represents the rate of production,

**k1** represents the speed of consumption,

**k2** represents cost of storing a unit of unsold good per unit of time,

**Y** represents potential demand,

**t** represents time,

**T** represents the end of the time interval

**p** represents the price,

$n_p$ represents the coefficient of the rate of sales of the consumer good.

However, in this model **(2.1)** the author did not consider the important role of the customer, who buys the manufacturer's product. **Paraev** in his recent paper **[7]** includes a customer in the model and assumes that the customer gets a profit from buying and reselling the product. What kind of profit? The customer can use the good as a supply for his own production activity or he can just resell it at the higher price. Then the customer can choose such a buying and reselling policy that maximizes his cumulative profit. Thus, both manufacturer and the customer (buyer) want to maximize their profits. We understand that their actions depend on each other. Therefore, we have a cooperative game situation. Let us see how Paraev modified model **(2.1)**:

$$\begin{cases} \dot{x}(t) = u(t) - P_1 \\ \dot{y}(t) = P_1 - P_2 \end{cases} \tag{2.2}$$

Here **u(t)** is the rate of production,

$P_1 = x(t)v(t)$ is the rate of purchasing the good.

$v(t)$ the coefficient of purchasing the good,

$P_2 = y(t)w(t)$ is the rate of reselling the good,

$w(t)$ the coefficient of reselling the good.

Paraev solved the two–criteria optimal control problem for model **(2.2)** and obtained the optimal solution. He found such solutions when both players have positive profit. Otherwise, one of the two will not "participate in the game".

However, his model **(2. 2)**, which describes a perishable good, is too simple. In fact, perishable goods can be spoiled either in the manufacturer's or in the retailer's storage, with rate of spoilage, $q$. This additional condition makes the problem more complex but also more realistic. Our model, which is a combination of models **(2.1)** and **(2.2)**, will be described in the following chapter.

# CHAPTER III

# MODEL

The model investigated in this paper will involve a manufacturer, which produces, stores, and sells a perishable consumer product, and a retailer, who buys the manufacturer's product and resells it to the third party.

We will find parameters of the model at which both a manufacturing company and the reseller obtain their maximum profit. By having both players with the same goal and using the same manufactured item, we can see the application of the cooperative game theory in effect. Both the manufacturer and the reseller depend on each other. The manufacturer needs someone to buy his product at a profit and the reseller needs a product he can resell, also for a profit. Of course, if the item is not manufactured, then the reseller cannot make his profit and the manufacturer will not have his factory without a product.

The consumer product in this model might be a food item or a medication, both of which have an allotted shelf life. Our consumer could have several of this product in his possession that would indicate that he does not need to purchase this item until his supply is consumed. The unconsumed product will inhibit the reseller from selling that item to that consumer. This could mean that the retailer will have the product in his possession and have to pay for storage. That would reduce the retailer's profit.

We are exploring our model as a cooperative game, which means that the manufacturing and retailing event must be a 'win-win' outcome. The manufacturer will want to produce the right amount of the product in order for him to be selling it at the lowest price. The reseller will want to sell the right amount in order for him to buy the product at the lowest price and then resell at a profit. What if the manufacturer makes one item a day for 5 days and sells each at a very high price. The reseller must buy at a high price and only has 5 items to resell at this high price. The reseller will only be selling to potentially 5 people, who are willing or wanting to pay this high amount.

Will the manufacturing plant be able to continue in business if it only produces 5 of this item a week? Will the utilities, taxes, lease, benefits, wages, profit sharing, and capital improvements be able to be realized from producing 5 items a week? From the reseller's point of view, if he has only five items a week to purchase from the manufacturer, does that supply enough items to resell in order to make enough profit to purchase more inventory ? The reseller will have some of the same expenses as the manufacturer: lease, utilities, taxes, benefits, wages, profit sharing, and also a much larger marketing expense. Will the 5 items produced each week be able to sustain each company?

Our model may be described as:

$$\textit{Model} \quad \begin{cases} \dot{x}(t) = -v(t)x(t) - q \cdot x(t) + u(t) \\ \dot{y}(t) = v(t)x(t) - w(t)y(t) - q \cdot y(t) \\ x(0) = x_0,\ y(0) = y_0, \quad x_0, y_0 > 0 \end{cases} \qquad (3.1)$$

$x(t)$ represents the amount of manufactured items at time t, and x(0) is its initial condition.

$y(t)$ represents the amount of retailer sales at time t, and y(0) is its initial condition.

$u(t)$ represents the rate of production of manufactured items per unit of time.

$v(t)$ represents the rate of purchasing a unit of the good by reseller from the manufacturer.

$w(t)$ represents the rate of reselling a unit of the good to the third party.

$c_1$ represents the amount the manufacturer sells his item for–manufacturer's price.

$c_2$ represents the amount the retailer sells the item for-retailer's price $k_1$ represents the cost of storage of 1 unit of the manufactured item; this amount is paid by the manufacturer on an item he has not sold to the reseller.

$k_2$ represents the cost of storage of 1 unit for the reseller; this amount is paid for by the reseller on an item he has not sold yet.

$q$ represents the spoilage coefficient; this is for items that the manufacturer or the reseller has not sold and that need to be destroyed or disposed of as they are expired.

$t$ represents time, $t \in [0, T]$

$T$ is the end of the whole planning period of time in which we want to maximize profit. The units for time can be days, weeks or months.

The main objective is to choose such controls **u(t)**- production rate, **v(t)**- rate of purchasing, and **w(t)**- rate of reselling, that will maximize profits for the manufacturer and the reseller. The manufacturing rate of production of items, rate of purchasing, and the rate of reselling will be our control parameters.

The rate of change of the manufacturer's profit can be written as

$\dot{z}_1 = c_1 v(t)x(t) - k_1 x(t) - u(t)$.

The first term is the revenue; the next two terms are costs, which are subtracted from the revenue. The manufacturer's revenue is the price multiplied by the rate of goods produced times the number of items manufactured. The manufacturer's costs are the storage costs per item and the cost of production.

The equation below is the rate of change of the retailer's profit:

$\dot{z}_2 = c_2 w(t)y(t) - c_1 v(t)x(t) - k_2 y(t)$.

The first term is the revenue; the next two terms are the costs, which are subtracted from the revenue.

Both parties want to choose such a strategy (optimal production plan, optimal buying and reselling plan, etc.) that will maximize their cumulative profits. The goal of the manufacturer is to maximize his profit: $\max \int_0^T \dot{z}_1 dt = \int_0^T (c_1 v(t)x(t) - k_1 x(t) - u(t))dt$.

$$z_1(T) - z_1(0) = \int_0^T (c_1 v(t)x(t) - k_1 x(t) - u(t))dt,$$

which is equivalent to maximizing the following functional:

$$J_1 = \int_0^T \big(c_1 v(t)x(t) - k_1 x(t) - u(t)\big)dt \to \max_{u(\cdot)\in D(T)} \tag{3.2}$$

The goal of the buyer is to maximize his cumulative profits or the following functional:

$$J_2 = \int_0^T \left(c_2 w(t) y(t) - c_1 v(t) x(t) - k_2 y(t)\right) dt \to \max_{w(\cdot), v(\cdot) \in D(T)} \tag{3.3}$$

Here **D(T)** is the control set, such that **D(T)** is the set of all Lebesgue[1] measurable functions **u(t)**, **v(t)**, and **w(t)** satisfying the following inequalities :

$$0 \le u(t) \le u_0,\ 0 \le v(t) \le v_0 \text{ and } 0 \le w(t) \le w_0. \tag{3.4}$$

Since all the control parameters are bounded functions, we will need to apply optimal control theory to solve the problems stated above. The problem can be classified as a"cooperative game problem" or "two criteria optimal control problem".

The **Main Problem** can be summarized as:

**For a manufacturer:** Select such control $u_*(t),\ t \in [0, T]$, that would maximize $J_1$.

**For the retailer or reseller:** Choose such controls $v_*(t),\ w_*(t),\ t \in [0, T]$ that would maximize functional $J_2$.

In order to solve it, we have to discuss the following:

If $u(t) = 0$, then $J_1 = J_2 = 0$ (the good is not produced)

If $u(t) \ne 0$, but $v(t) = 0$ (the good is not purchased), then $J_1 < 0,\ J_2 = 0$.

We say that the solution to the main problem exists if and only if $J_1 > 0,\ J_2 > 0$.

Therefore, first we need to solve two auxiliary problems in order to find control variables that will maximize $J_2$ and $J_1$.

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[1] Lebesgue measure is a piecewise continuous function in which we assume the existence of the finite limits at a point of discontinuity.

**Auxiliary Problem 1** can be summarized as: if $\mathbf{v(t) \neq 0}$ is given, find such a function

$\overline{\boldsymbol{u}}(\boldsymbol{t}), \boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}]$, at which functional $\boldsymbol{J}_1$ is maximum.

**Auxiliary Problem 2** can be stated as: If $\mathbf{u(t) \neq 0}$ is given, find such functions,

$\overline{\boldsymbol{v}}(\boldsymbol{t}), \overline{\boldsymbol{w}}(\boldsymbol{t}), \boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}]$, for which functional $\boldsymbol{J}_2$ is maximum.

Since each functional is linear in control and each control is compact, a solution for both auxiliary problems will exist. (See **Lee** and **Markus [8]** or **[2])**

The customer has no control over the manufacturer's production rates just as the manufacturer has no control over the decision to purchase by the consumer. The optimum production rate would allow the price to be attractive for the consumer and benefit both of the cooperative game players. This, in turn, would positively influence the consumer to purchase, leading back to a benefit for the other cooperative player, the manufacturer. The cycle could continue to the mutual benefit of both parties.

## CHAPTER IV

## PROPERTIES OF THE MODEL

In order to solve optimal control problems stated in the previous section, we need to investigate properties of the variables of our system.

<u>Theorem 1</u>. ***Let*** $u(\cdot)$, $v(\cdot)$, ***and*** $w(\cdot) \in D(T)$ ***be some control functions. Then there exists a solution*** $\left(x(t), y(t)\right)$ ***of system* (3.1) *corresponding to all controls defined on the closed interval* [0, T], *that satisfy the following inequalities:***

$$x(t) > 0,\ y(t) > 0,\ t \in \left[0, T\right] \tag{4.1}$$

Proof part 1: Let us consider the first equation of system **(3.1)**:

$$\dot{x}(t) = -(v(t) + q)x(t) + u(t) \quad x(0) = x_0, x_0 > 0$$

Using the method of a parameter variation **[9]**, first we will solve the homogeneous equation.

Let $\mu(t) = v(t) + q$. Then we can rewrite equation **(3.1)** as:

$$\frac{dx}{dt} = -\mu(t) \cdot x(t) + u(t) \tag{4.2}$$

$$\frac{dx}{x} = -\mu(t)dt$$

$$\ln|x| = -\int \mu(t)dt + \ln c$$

$$|x| = c e^{-\int \mu(t)dt}$$

$$x(t) = C(t) \cdot e^{-\int \mu(t)dt} \tag{4.3}$$

$$x'(t) = C'(t) \cdot e^{-\int \mu(t)dt} + C(t) \cdot (-\mu(t)) e^{-\int \mu(t)dt} \tag{4.4}$$

Now substitute equations **(4.2)** and **(4.3)** into equation **(4.4)**, resulting in the following:

$$C'(t) \cdot e^{-\int \mu(t)dt} + C(t) \cdot (-\mu(t)) e^{-\int \mu(t)dt} = -\mu(t) \cdot C(t) \cdot e^{-\int \mu(t)dt} + u(t) \tag{4.5}$$

After reducing the equation we are left with: $C'(t) \cdot e^{-\int \mu(t)dt} = u(t)$. Multiplying both sides of the equation by $e^{\int \mu(t)dt}$, we obtain

$$C'(t) = u(t) \cdot e^{\int \mu(t)dt}$$

$$C(t) = \int u(t) \cdot e^{\int \mu(t)dt} dt + c_0$$

$$x(t) = \left( x_0 + \int_0^t e^{qs + \int_0^s v(\xi)d\xi} u(s)ds \right) \cdot e^{-qt - \int_0^t v(\xi)d\xi}$$

Now we prove that $x(t) > 0$, with another method **[9].**

$$x'(t) = -\mu(t)x(t) + u(t)$$

$$x'(t) + \mu(t)x(t) = u(t)$$

Multiply each side by integrating factor.

$$\frac{d}{dt}\left( e^{qt + \int_0^t v(\xi)d\xi} x(t) \right) = e^{qt + \int_0^t v(\xi)d\xi} u(t)$$

$$e^{qt+\int_0^t v(\xi)d\xi}x(t)-x_0=\int_0^t e^{qs+\int_0^s v(\xi)d\xi}u(s)ds$$

$$x(t)=e^{-qt-\int_0^t v(\xi)d\xi}\left(x_0+\int_0^t e^{qs+\int_0^s v(\xi)d\xi}\right)\forall\ u(\cdot),v(\cdot)\quad x(t)>0,\quad t\in[0,T]$$

Part 2: Let us prove that $y(t)>0$. Consider taking the second equation of system **(3.1)**.

$$\begin{cases}\dot{y}(t)=-(w(t+q))y(t)+v(t)\cdot x(t)\\ y(0)=y_0\ ,\ \ y_0>0\end{cases}$$

$$\dot{y}(t)+(w(t)+q)\,y(t)=v(t)x(t)$$

$$\frac{d}{dt}\left(e^{\int(w(t)+q)dt}\cdot y(t)\right)=e^{\int(w(t)+q)dt}v(t)x(t)$$

$$\frac{d}{dt}\left(e^{q(t)+\int_0^t w(\xi)d\xi}\cdot y(t)\right)=e^{qt+\int_0^t w(\xi)d\xi}v(t)x(t)$$

$$e^{qt+\int_0^t w(\xi)d\xi}y(t)-e^{q\cdot 0\int_0^0 w(\xi)d\xi}\cdot y(0)=\int_0^t e^{qs+\int_0^s w(\xi)d\xi}\cdot v(s)x(s)ds$$

$$e^{qt+\int_0^t w(\xi)d\xi\cdot}y(t)-y(0)=\int_0^t e^{qs+\int_0^s w(\xi)d(\xi)}\cdot v(s)x(s)ds$$

$$e^{qt+\int_0^t w(\xi)d\xi}\cdot y(t)=\int_0^t e^{qs+\int_0^s w(\xi)d\xi}\cdot v(s)x(s)ds+y(0)$$

$$y(t)=e^{-qt-\int_0^t w(\xi)d\xi}\cdot\left(y(0)+\int_0^t e^{qs+\int_0^s w(\xi)d\xi}\cdot v(s)x(s)ds\right)$$

$$y(t) = e^{-qt-\int_0^t w(\xi)d\xi}\left(y(0) + \int_0^t e^{qs+\int_0^s w(\xi)d(\xi)} \cdot v(s)x(s)d(s)\right)$$

Therefore, $x(t) > 0$, $y(t) > 0\ \forall\ u(\cdot), v(\cdot)$ and $w(\cdot)\ t \in [0, T]$. The theorem is proven.□

## CHAPTER V

## SOLVING OPTIMAL CONTROL PROBLEM FOR A RETAILER

Our main problem was for the manufacturer and reseller to have their profits maximized. For $J_1$ and $J_2$ to have the highest profits, we need to choose control variables for the manufacturer and for the retailer that will allow $J_1$ and $J_2$ to approach maximums. We will first solve our two auxiliary problems in order to find values of our control variables. First we will maximize $J_2$.

### 5.1 Hamiltonian and Adjoint System

Let us solve **Problem 2** stated in chapter III. Find values of control variables $\bar{v}(t), \bar{w}(t)$, so that $J_2$ is maximized. Let $(\bar{x}(t), \bar{y}(t))$ be the optimal trajectory and $(\bar{v}(t), \bar{w}(t))$ be the corresponding optimal controls. Applying the Pontryagin Maximum Principle, we will obtain the following Hamiltonian:

$$H_2 = \psi_1(t) \cdot \dot{x}(t) + \psi_2(t) \cdot \dot{y}(t) + \dot{J}_2(t)$$

Here $\psi_1(t)$ and $\psi_2(t)$ are adjoint variables.

Then replacing $\dot{x}$ and $\dot{y}$ from system **(3.1)** and $\dot{J}_2(t)$ from **(3.3),**

$H_2$ can be rewritten as :

$$H_2 = (-vx - qx + u)\psi_1 + (vx - wy - qy)\psi_2 + (c_2wy - c_1vx - k_2y) \tag{5.1}$$

or in the form **(5.2)** below:

$$H_2 = v(t)[-x(t)\cdot\psi_1(t)+x(t)\cdot\psi_2(t)-c_1x(t)]+w(t)[-y(t)\psi_2(t)+c_2y(t)]$$

$$-qx(t)\cdot\psi_1(t)-qy(t)\cdot\psi_2(t)-k_2y(t) \qquad \textbf{(5.2)}$$

Obviously, $H_2$ is linear in both controls, **v(t)** and **w(t),** and control set **D(T)** is a compact set. Then the solution to the optimal control **Problem 2** exists. **[2, 8]**

By definition:

$$\dot{\psi}_1(t) = \frac{-\partial H_2}{\partial x}$$

$$\dot{\psi}_2(t) = \frac{-\partial H_2}{\partial y},$$

then differentiating **(5.1):**

$$\frac{\partial H_2}{\partial x} = -v\psi_1 + v\psi_2 - q\psi_1 - c_1 v$$

$$\frac{\partial H_2}{\partial y} = -w\psi_2 + q\psi_2 + c_2 w - k_2,$$

and denoting $\bar{v}, \bar{w}$ as optimal controls, we obtain the adjoint system:

$$\begin{cases} \dot{\psi}_1(t) = \bar{v}(t)\psi_1(t) - \bar{v}(t)\psi_2(t) + q\psi_1(t) + c_1\bar{v}(t) \\ \dot{\psi}_2(t) = \bar{w}(t)\psi_2(t) + q\psi_2(t) - c_2\bar{w}(t) + k_2 \\ \psi_1(T) = 0, \quad \psi_2(T) = 0 \end{cases} \qquad \textbf{(5.3)}$$

and introduce switching functions as:

$$\tilde{L}_v(t) = -x(t)[\psi_1(t) - \psi_2(t) + c_1]$$

$$\tilde{L}_w(t) = -y(t)[\psi_2(t) - c_2]$$

Now **(5.2)** can also be rewritten as :

$$\bar{v}(t) \cdot \tilde{L}_v(t) + w(t)\tilde{L}_w(t) - q[x(t) \cdot \psi_1(t) + y(t) \cdot \psi_2(t)] - k_2 y(t)$$

Since $x(t) > 0$ and $y(t) > 0$, (stated by **Theorem 1**) we can use new switching functions:

$$L_v(t) = -[\psi_1(t) - \psi_2(t) + c_1]$$

$$L_w(t) = -[\psi_2(t) - c_2] \qquad \textbf{(5.4)}$$

Switching functions determine the behavior of optimal controls. It follows from the Pontryagin Maximum Principal that the optimal controls $\bar{v}(t)$ and $\bar{w}(t)$ can be written as:

$$\bar{v}(t) = \begin{cases} v_0, & L_v(t) > 0 \\ n \in [0, v_0] & L_v(t) = 0 \\ 0 \;, & L_v(t) < 0 \end{cases} \qquad \textbf{(5.5)}$$

$$\bar{w}(t) = \begin{cases} w_0, & L_w(t) > 0 \\ n \in [0, w_0] & L_w(t) = 0 \\ 0 \;, & L_w(t) < 0 \end{cases} \qquad \textbf{(5.6)}$$

If $L_v(t) > 0$, then $\bar{v}(t)$ must take its maximum value, that is, $v_0$. If $L_v(t) = 0$, then $\bar{v}(t)$ can take any value from $[0, v_0]$ (this is a case of singular control). If $L_v(t) < 0$, then $\bar{v}(t)$ takes its minimum value, that is, $\bar{v} = 0$. To illustrate how switching function $L_v(t)$ might influence optimal control $\bar{v}(t)$, an example is provided below.

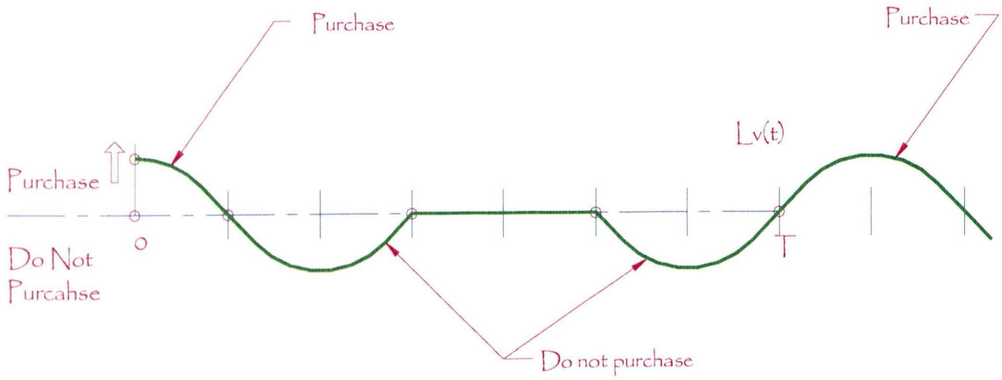


Figure 5.1 Switching Function $L_v(t)$

Therefore, we need to investigate the behavior of the functions of switching ($L_v(t)$ and $L_w(t)$). Next, using the adjoint system **(5.3)** we will obtain the system of differential equations for $L_v(t)$ and $L_w(t)$ using a similar approach to that used in **[5]**.

The adjoint system **(5.3)** can be rewritten as:

$$\begin{cases} \dot{\psi}_1 = -v(t) \cdot L_v(t) + q\psi_1(t) \\ \dot{\psi}_2 = -w(t) \cdot L_w(t) + q\psi_2(t) + k_2 \end{cases}$$

Let us find the following derivative:

$$(\psi_1 - \psi_2 - c_1)' = \dot{\psi}_1 - \dot{\psi}_2$$

$$-\dot{L}_v(t) = -v(t) \cdot L_v(t) + w(t) \cdot L_w(t) + q(\psi_1(t) - \psi_2(t)) - k_2$$

The goal is to get the equation for the switching function.

In order to find the boundary conditions for $L_v(t)$ and $L_w(t)$ ,we will replace **t** by **T**. Since

$\psi_1(T) = \psi_2(T) = 0$, then

$L_v(T) = -(\psi_1(T) - \psi_2(T) + C_1) = -C_1$

$L_w(T) = -(\psi_2(T) - C_2) = C_2$

Now we have a system for switching functions. (This is a two point boundary value problem for the functions of switching. **[10]**)

$$\begin{cases} \dot{L}_v(t) = (\bar{v}(t) + q)L_v(t) - \bar{w}(t)L_w(t) + (qc_1 + k_2) \\ \dot{L}_w(t) = (\bar{w}(t) + q)L_w(t) - (qc_2 + k_2) \\ L_v(T) = -c_1, \quad L_w(T) = c_2 \end{cases} \tag{5.7}$$

Let us consider the second equation from **(5.7)**:

$$\dot{L}_w(t) = (\bar{w}(t) + q)L_w(t) - (qc_2 + k_2)$$

Since this equation is linear in $L_w(t)$, we can state the following Lemma.

__Lemma 1__. ***The switching function $L_w(t) \neq 0$ on any nonzero subinterval $\Delta \in [0,T]$.***

Proof. We will prove it by contradiction. Let $L_w(t) \equiv 0$ on some $\Delta \in [0,T]$. Recall the expression for $L_w(t)$: $\dot{L}_w = (\overline{w}(t) + q)L_w(t) - (qc_2 + k_2)$.

If $L_w(t) \equiv 0$, then $\dot{L}_w(t) \equiv 0$ in $\Delta$. $0 = -(qc_2 + k_2)$ and $-(qc_2 + k_2) < 0$. This is a false statement and hence we obtain the contradiction. The **Lemma 1** is proven. □

Since it follows from Lemma 1 that $L_w(t)$ does not contain any segment $\Delta$ on which $L_w(t) = 0$ (see figure 5.2), then optimal control $\overline{w}(t)$ does not have singular segments and $\overline{w}(t)$ is a piecewise constant function **[11]**, and **(5.6)** can be rewritten as:

$$\overline{w}(t) = \begin{cases} w_o, & L_w(t) > 0 \\ 0, & L_w(t) < 0 \end{cases} \tag{5.8}$$

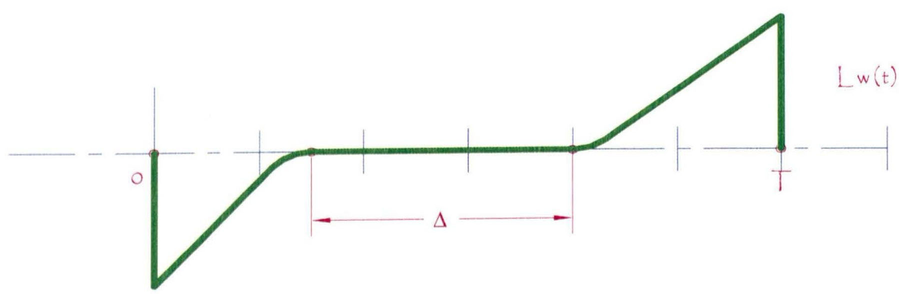


Figure 5.2 Switching Function $L_w(t)$

Though $L_w(t)$ cannot be zero on any nonzero finite interval, it can be zero at some points of $[0,T]$ (see Figure 5.3 below).

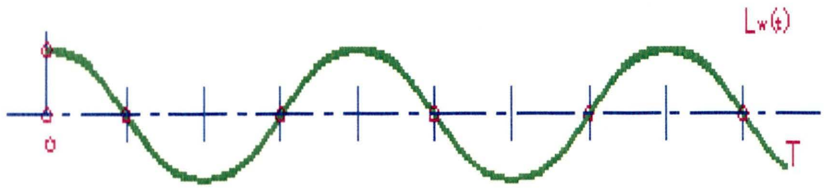


Figure 5.3 Sign Change of $L_w(t)$

In order to find optimal control $\overline{w(t)}$ given by **(5.8)**, we need to know how many times $L_w(t)$ changes its sign on $[0,T]$.

Now we can state the following:

<u>*Lemma 2.*</u> ***$L_w(t)$ has no switchings on $[0,T]$.***

Proof. At moment **t=0** let $L_w(0)>0$. Then control $w(0)=w_0$. Also assume that $t=\tau$ (switching time), $L_w(\tau)=0$. Then at the moment of switching , $t=\tau$ :

$$\dot{L}_w(\tau) = -(qc_2 + k_2) < 0$$

i.e., $L_w(t)$ is monotonically decreasing and cannot change its sign any more, so $L_w(T)$ must be negative . On the other hand, looking at the boundary condition, $L_w(T)=c_2>0$, such a situation is not possible. Therefore, if $L_w(0)>0$, then switching cannot occur and there is no switching.

Let us assume that $\boldsymbol{L_w(0) < 0}$ and that switching occurs at $\boldsymbol{t = \tau}$; i.e., $\boldsymbol{L_w(\tau) = 0}$, and

$\boldsymbol{\dot{L}_w(\tau) = -(qc_2 + k_2) < 0}$. Again $\boldsymbol{\dot{L}_w(\tau) < 0 \Rightarrow L_w(t)}$ must decrease in the entire interval.

This case must be eliminated as impossible because of the boundary condition. □

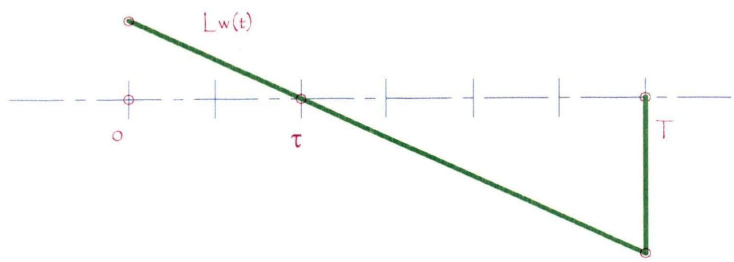


Figure 5.4 One Zero of $L_w(t)$

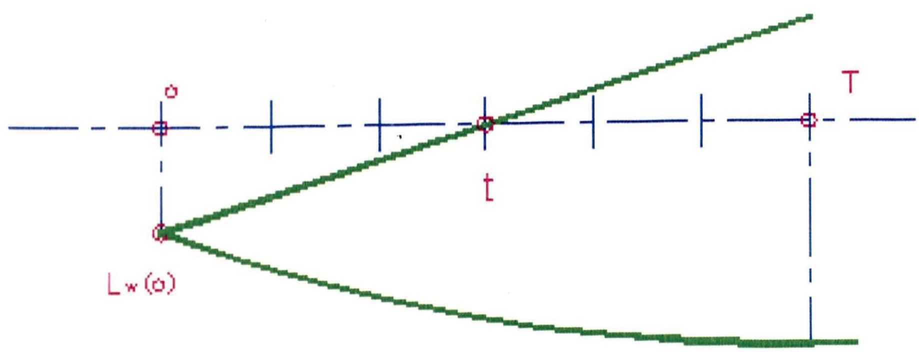


Figure 5.5 Switching at $\boldsymbol{\tau}$

Therefore, $\boldsymbol{L_w(t) > 0}$ for any $\boldsymbol{t \in [0,T]}$, and can be illustrated by the graph below.

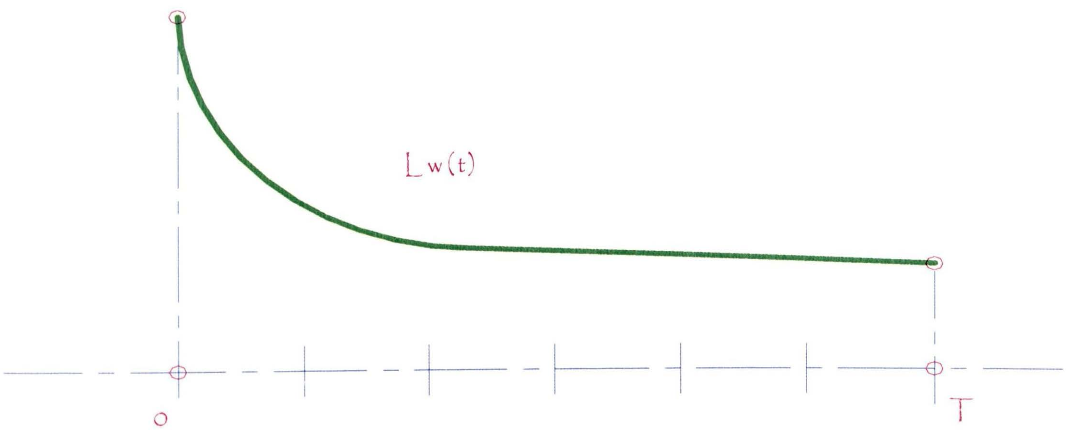


Figure 5.6 Switching Function $L_w(t)$

Next, the optimal control $\boldsymbol{\overline{w}(t)}$ is:

$$\boldsymbol{\overline{w}(t) = w_0,\ \ t \in [0,T]}. \tag{5.9}$$

The retailer must resell at the same rate during the entire time interval.

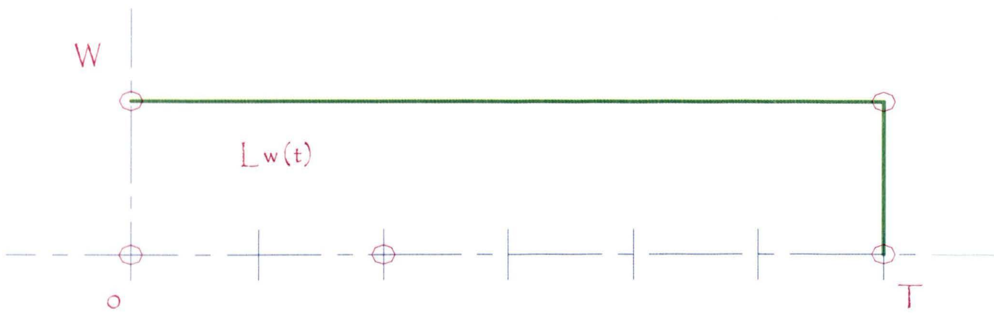


Figure 5.7 $\overline{w}(t) = w_0$ Constant positive function

## 5.2 Finding Switching Function $L_w(t)$ Analytically

We will now solve the second equation from **(5.7)**:

$$\dot{L}_w(t) = (\overline{w}(t) + q)L_w(t) - (qc_2 + k_2)$$

$$L_w(T) = c_2$$

Using equation **(5.9)**, it can be rewritten as $\dot{L}_w(t) = (w_0 + q)L_w(t) - (qc_2 + k_{2)})$.

$$\frac{d}{dt}\left(e^{-(w_0+q)t}L_w(t)\right) = -(qc_2 + k_2)e^{-(w_0+q)t}$$

$$e^{-(w_0+q)T}c_2 - e^{-(w_0+q)t}L_w(t) = -(qc_2+k_2)\int_t^T e^{-(w_0+q)s}ds$$

$$e^{-(w_0+q)(T-t)}c_2 - \frac{qc_2+k_2}{w_0+q}e^{-(w_0+q)(T-t)} + \frac{qc_2+k_2}{w_0+q} = L_w(t)$$

$$L_w(t) = \frac{qc_2+k_2}{w_0+q} + \left(c_2 - \frac{qc_2+k_2}{w_0+q}\right)e^{-(w_0+q)(T-t)}$$

$$L_w(t) = \frac{c_2q+k_2}{w_0+q} + \frac{c_2w_0-k_2}{w_0+q}e^{-(w_0+q)(T-t)} \tag{5.10}$$

Notice that $L_w(T) = c_2$.

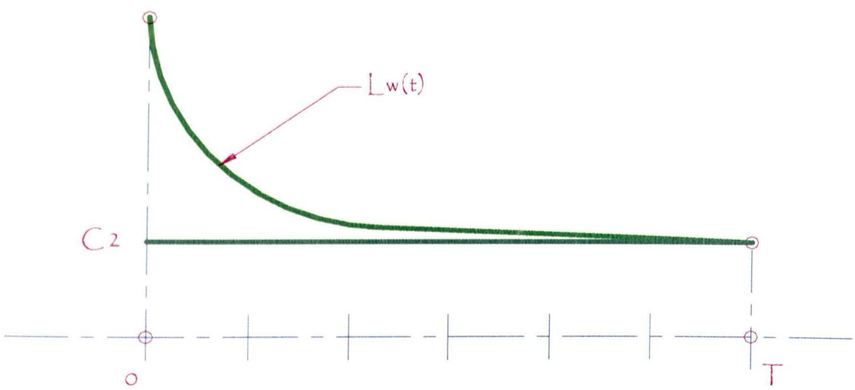


Figure 5.8 $L_w(t) > 0$ and $C_2$ = Positive Constant

### 5.3 Finding Switching Function $L_v(t)$ and Optimal Control $\bar{v}(t)$

Next we will study $L_v(t)$ and optimal control $\bar{v}(t)$. We will consider two cases.

**Case 1**: $c_2 w_0 - k_2 = 0$

Then from **(5.10)** we obtain that $L_w(t) = \dfrac{c_2 q + k_2}{w_0 + q}$

$$L_w(t) = \frac{c_2 q + c_2 w_0}{w_0 + q} = \frac{c_2(q + w_0)}{w_0 + q} = c_2 = L_w(t).$$

Next we can rewrite the first equation in **(5.7)** as :

$$\dot{L}_v(t) = (\bar{v}(t) + q)L_v(t) - c_2 w_0 + (qc_1 + c_2 w_0) \quad \text{or}$$

$$\dot{L}_v(t) = (\bar{v}(t) + q)L_v(t) + qc_1$$

Thus we obtain the boundary Cauchy problem **[12]** for $L_v(t)$:

Therefore, $\begin{cases} \dot{L}_v(t) = (\bar{v}(t) + q)L_v(t) + qc_1 \\ L_v(t) = -c_1 \end{cases}$

Next we can state:

<u>**Lemma 3**</u>. $L_v(t)$ *cannot be zero on any finite subinterval* $\Delta \in [0, T]$.

Proof. We will prove it by contradiction. Let $L_v(t) \equiv 0$ on some interval $\Delta \in [0, T]$:

$L_v(t) \equiv 0 \Rightarrow \dot{L}_v(t) \equiv 0$ in $\Delta$ .

$0 = 0 + qc_1 > 0$

However, $0 \neq (qc_1 + k_2)$. This is a false statement and hence we obtain the contradiction.

The lemma is proven.□

The optimal control $\bar{v}(t)$ is a piecewise constant function, such that

$$\bar{v}(t) = \begin{cases} v_0, & L_v(t) > 0 \\ 0, & L_v(t) < 0 \end{cases}.$$

<u>Proposition</u> 1. *If* $c_2 w_0 - k_2 = 0$, *then* $L_v(t)$ *does not have any zero on* $[0,T]$.

Proof: At moment $t = 0$, let $L_v(0) > 0$. Then control $v(0) = v_0$, so let $t = \tau$ (switching time ), $L_v(\tau) = 0$ at the moment of switching

$$\dot{L}_v(\tau) = (\bar{v}(\tau) + q)L_v(\tau) + qc_1 \tag{5.11}$$

$$\dot{L}_v(\tau) = qc_1 \Rightarrow \dot{L}_v(\tau) > 0 .$$

Looking at boundary condition $L_v(t) = -c_2 < 0$, such a case is not possible. Therefore, $L_v(0) > 0$ cannot occur according to the boundary condition.

Next, we assume that at $t = 0$, $L_v(0) < 0$. Then control $v(0) = 0$. If $t = \tau$ is the time of switching, then again $L_v(\tau) = 0$ at the moment of switching, and relationships **(5.11)** hold. Again this case cannot occur because of the boundary condition $L_v(T) = -c_1 < 0$. Therefore, in **case 1**, the switching function $L_v(t) < 0$ on $[0,T]$ and therefore the optimal control $\bar{v}(t)$ is $\bar{v}(t) \equiv 0$ on $[0,T]$. □

Therefore, if $c_2 w_0 - k_2 = 0$ (**case 1**) or $k_2 = c_2 w_0$ on $[0,T]$, then $\bar{v}(t) \equiv 0$ and as it stated before $\bar{w} = w_0$ on $[0,T]$. (The retailer does not purchase but sells only.)

<u>Case 2:</u> $c_2 w_0 - k_2 \neq 0$

$$\begin{cases} \dot{L}_v(t) = (\bar{v}(t) + q) L_v(t) - P(t) \\ L_v(t) = -c_1 \end{cases} \tag{5.12}$$

where $P(t) = \wp(c_2 w_0 - k_2) e^{-(w_0+q)(T-t)} + \wp(qc_2 + k_2) - (qc_1 + k_2)$ and **(5.13)**

$$\wp = \frac{w_0}{w_0 + q} < 1 \tag{5.14}$$

Notice that $P(t)$ has at most one zero (as exponential, monotonically decreasing function)

**<u>Proposition 2</u>:** ***In* case 2 $L_v(t)$ *has at most two zeros.***

Proof: We will prove it by contradiction. Let us assume that $0 < \theta_1 < \theta_2 < \theta_3 < T$ are three zeros of $L_v(t)$. $L_v(t)$ has at most two zeros. Integrating equation **(5.12)** we have

$$\frac{d}{dt}\left( e^{-\int_0^t (v(\xi)+q)d\xi} \cdot L_v(t) \right) = -e^{-\int_0^t (v(\xi)+q)d\xi} \cdot P(t) \tag{5.15}$$

which can be rewritten as

$$\frac{d}{dt} f(t) = \quad h(t).$$

If $\theta_i$ is a zero of $L_v(t)$, then it will also be a zero of function $f(t) = e^{-qt-\int_0^t v(\xi)d\xi} \cdot L_v(t)$.

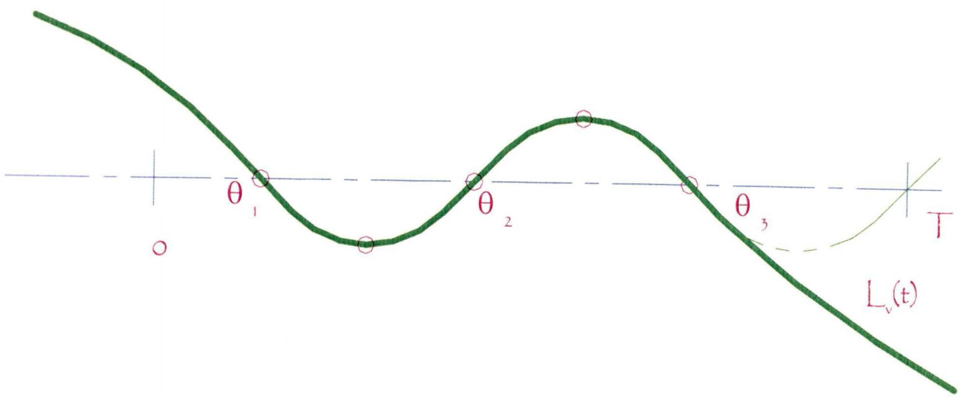


Figure 5.9 Three zeros

$$e^{qt-\int_0^t v(\xi)d\xi} \cdot L_v(t) = f(t)$$

Also from Rolle's Theorem **[13]** we know that if a function is continuous and differentiable on (0, T), then there is always at least one zero of the derivative between two zeros of the function.

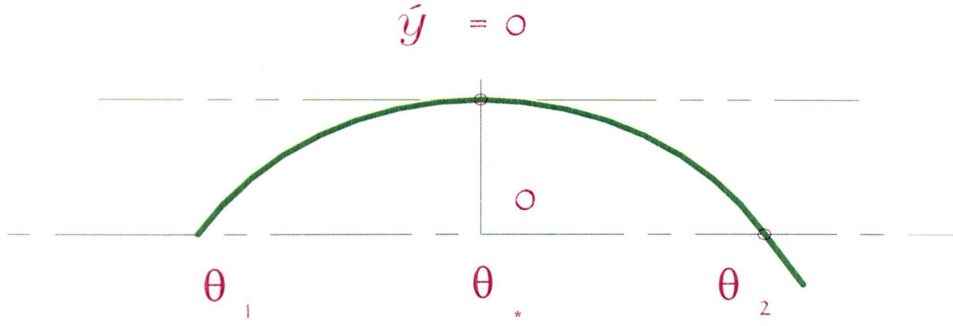


Figure 5.10 Rolle's theorem

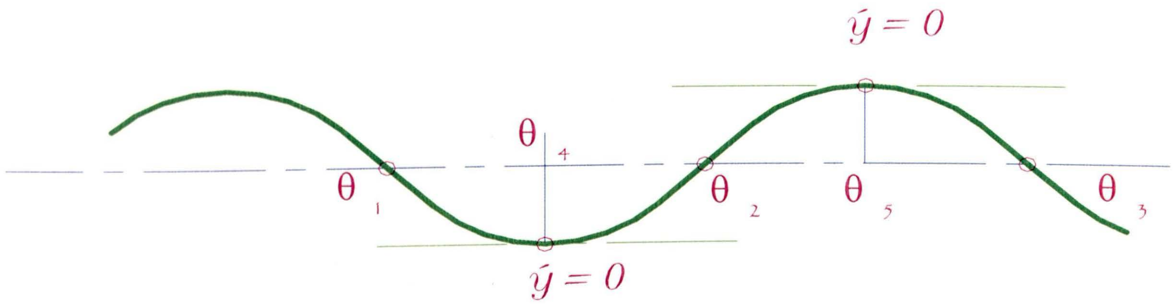


Figure 5.11 Rolle's theorem applied to $L_v(t)$.

Since $h(t)$ is the derivative of $f(t)$, $h(t)$ must have at least 2 zeros at $\theta_4$ and $\theta_5$ on $[0,T]$.

$(\theta_4 \in [\theta_1,\theta_2],\ \theta_5 \in [\theta_2,\theta_3])$

However, $h(t) = -e^{-qt-\int v(\xi)d\xi} \cdot P(t)$ has at most one zero. We obtain the contradiction and the statement is proven. □

Since $L_v(t)$ has at most 2 zeros and satisfies **(5.12)**, then $\bar{v}(t)$ can have one of the following forms:

$\bar{v}(t) =$

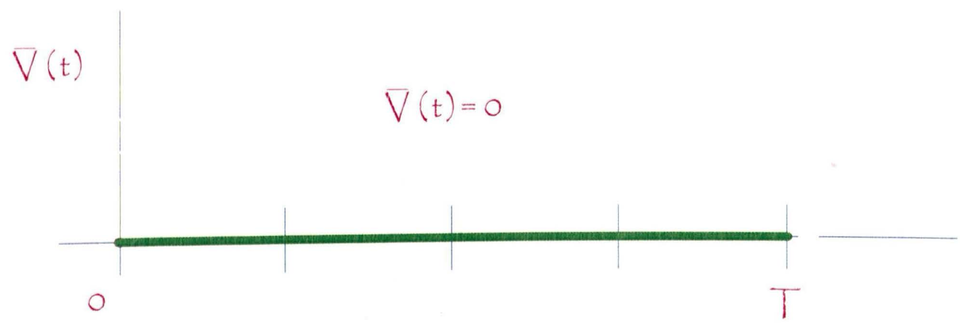


Figure 5.12 $\bar{v}(t) = 0$ No switching

$\bar{v}(t) =$

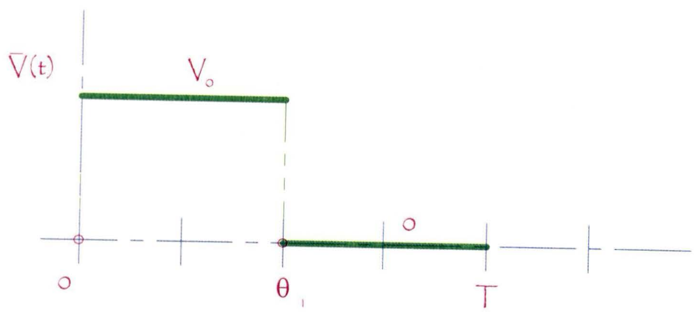


Figure 5.13. One Switching of **v(t)**

$\bar{v}(t)=$

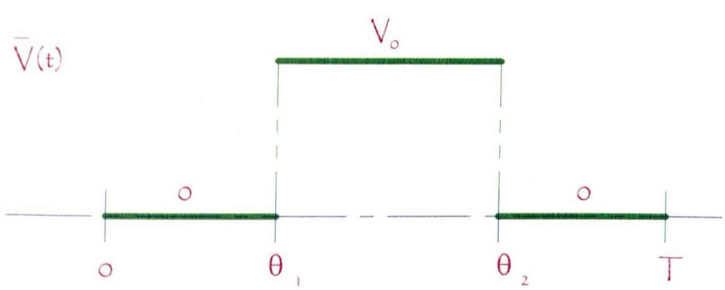


Figure 5.14 Two Switchings

Notice, that $\bar{v}(t)$ always ends in **0**, because of the boundary condition $L_v(T) = -c_1 < 0$.

<u>Lemma 4.</u> ***Switching function has at most one zero in*** **[0, T].**

Proof. Let us assume that $\theta \in (0,T)$ is the last switching in $\bar{v}(t)$, and $\bar{v}(t) = 0$ on $(\theta, T)$.

Then equation **(5.12)** on $(\theta, T)$ will be rewritten as :

$$\begin{cases} \dot{L}_v(t) = q \cdot L_v(t) - P(t) \\ L_v(t) = -c_1 \end{cases}$$

$$\frac{d}{dt}\left(e^{-qt} \cdot L_v(t)\right) = -e^{-qt} \cdot P(t)$$

Integrating between $t$ and $T$ we obtain:

$$e^{-qT} \cdot L_v(T) - e^{-qt} \cdot L_v(t) = -\int_t^T e^{-qs} P(s)ds$$

$$-e^{-qT} \cdot c_1 - e^{-qt} \cdot L_v(t) = -\int_t^T e^{-qs} P(s)ds$$

$$-e^{-q(T-t)} \cdot c_1 - L_v(t) = -e^{qt} \cdot \int_t^T e^{-qs} P(s)ds$$

Substituting $P(t)$ from **(5.13)** we obtain:

$$L_v(t) = -c_1 e^{-q(T-t)} + e^{qt} \int_t^T e^{-qs} \left\{\wp(c_2 w_0 - k_2) e^{-(w_0+q)(T-s)} + \wp(qc_2 + k_2) - (qc_1 + k_2)\right\} ds$$

which after integration gives us the following:

$$L_v(t) = -\wp(c_2 - \frac{k_2}{w_0}) e^{-(w_0+q)(T-t)} + \left(\left(\wp(c_2 + \frac{k_2}{q}\right) - \left(c_1 + \frac{k_2}{q}\right)\right) = \wp$$

Replacing $\wp$ from **(5.14),** we have

$$\frac{-w_0c_2}{w_0+q}e^{-(w_0+q)(T-t)}+\frac{k_2}{w_0+q}e^{-(w_0+q)(T-t)}+(\wp-1)\frac{k_2}{q}+\frac{w_0c_2}{w_0+q}=L_v(t)$$

$$L_v(t)=-\frac{(w_0c_2-k_2)}{w_0+q}e^{-(w_0+q)(T-t)}+(\wp-1)\frac{k_2}{q}+\frac{w_0c_2}{w_0+q}-c_1$$

Using $\wp$ from **(5.14)** we will simplify the last three terms above:

$$\left(\frac{w_0}{w_0+q}-1\right)\frac{k_2}{q}+\frac{w_0c_2}{w_0+q}-c_1=\frac{(w_0-w_0-q)k_2}{(w_0+q)q}+\frac{w_0c_2}{w_0+q}-c_1=\frac{-k_2q+c_2w_0}{w_0+q}-c_1$$

Substituting the new expression back into the expression for $L_v(t)$, we find

$$L_v(t)=-\frac{(c_2w_0-k_2)}{(w_0+q)}e^{-(w_0+q)(T-t)}+\frac{c_2w_0-k_2q}{w_0+q}-c_1$$

or simplifying further,

$$L_v(t)=\left(\frac{c_2w_0-k_2}{w_0+q}\right)\left(1-e^{-(w_0+q)(T-t)}\right)-c_1 \tag{5.15}$$

We see that **(5.15)** satisfies the boundary condition from **(5.12)**: $(L_v(T)=-c_1)$

Moreover, function **(5.15)** is exponential and has at most one switching on $[0,T]$. □

### 5.4. Case 2. Finding the Moment of Switching of $L_v(t)$

Let $t=\theta$ be the moment of switching on $[0,T]$. Then $L_v(\theta)=0$. Replacing $t$ by $\theta$ in **(5.15)** we can find the moment of switching:

$$L_v(\theta)=\left(\frac{c_2w_0-k_2q}{w_0+q}\right)\left(1-e^{-(w_0+q)(T-\theta)}\right)-c_1=0$$

$$\frac{c_2w_0-k_2q}{w_0+q}-c_1=e^{-(w_0+q)(T-\theta)}\cdot\frac{(c_2w_0-k_2q)}{(w_0+q)}$$

$$c_2 w_0 - k_2 - c_1 w_0 - c_1 q = (c_2 w_0 - k_2 q) e^{-(w_0+q)(T-\theta)}$$

$$e^{(w_0+q)(\theta-T)} = \frac{c_2 w_0 - k_2 - c_1 w_0 - c_1 q}{c_2 w_0 - k_2}$$

$$e^{(w_0+q)(\theta-T)} = 1 - \frac{c_1 (w_0 + q)}{c_2 w_0 - k_2} \tag{5.16}$$

$$\theta = \frac{1}{(w_0 + q)} \ln\left(1 - \frac{c_1 (w_0 + q)}{c_2 w_0 - k_2}\right) + T \tag{5.17}$$

We understand that in order for **(5.17)** to be valid, some restrictions on the value of the parameters must be imposed. We can see from **(5.16)** that if $c_2 w_0 - k_2 > 0$ , and

$c_1 \cdot \frac{(w_0 + q)}{c_2 w_0 - k_2} \geq 1$, then $L_v(t)$ has no switchings. Therefore, if

$$c_2 w_0 - k_2 > 0 \text{ and } c_1 \geq \frac{c_2 w_0 - k_2}{w_0 + q}. \tag{5.18}$$

function $L_v(t) < 0$ for any $t \in [0, T]$. Then $\bar{v}(t) \equiv 0$ on the entire time interval $[0, T]$. If

$$c_2 w_0 - k_2 < 0, \tag{5.19}$$

then also $L_v(t) < 0$ for any $t \in [0, T]$ and $\bar{v}(t) \equiv 0 \quad \forall\, t \in [0, T]$.

In all cases different from **(5.18)** and **(5.19),** there is a switching on $[0, T]$!

Let us assume that $t = \theta$ is the switching of function $L_v(t)$ on $[0, T]$. Thus, the moment of switching $\theta$ must be less than $T$ and greater than $0$. If $\theta < T$, then the first term of **(5.17)** must be negative, which can happen only if $0 < 1 - \frac{c_1 (w_0 + q)}{c_2 w_0 - k_2} < 1$ or $\frac{c_1 (w_0 + q)}{c_2 w_0 - k_2} > 0$.

Then $c_2 w_0 - k_2 > 0$.

If $\theta > 0$, then $\dfrac{1}{w_0 + q} \ln\left(1 - \dfrac{c_1(w_0 + q)}{c_2 w_0 - k_2}\right) + T > 0$

$$\ln\left(1 - \frac{c_1(w_0 + q)}{c_2 w_0 - k_2}\right) > -T \cdot (w_0 + q)$$

$$1 - \frac{c_1(w_0 + q)}{c_2 w_0 - k_2} > e^{-T(w_0 + q)}$$

$$\frac{c_1(w_0 + q)}{c_2 w_0 - k_2} < 1 - e^{-T(w_0 + q)} \tag{5.20}$$

### 5.5 Solution for Problems 1 and 2 and the Main Problem

First, let us generalize our results for **Problem 2.**

1. Optimal control $\overline{w(t)}$ is always constant and takes the value $w_0$ in $[0,\ T]$; that is,

$$\overline{w(t)} = w_0, \qquad t \in [0, T]. \tag{5.21}$$

2. Optimal control $\overline{v(t)}$ may have no switchings or 1 switching.

a) If $w_0 - \dfrac{k_2}{c_2} \le 0$ or $\begin{cases} w_0 - \dfrac{k_2}{c_2} > 0 \\ \dfrac{c_1}{c_2} \cdot \dfrac{w_0 + q}{w_0 - \dfrac{k_2}{c_2}} \ge 1 - e^{-(w_0 + q)T} \end{cases}$ (5.22)

$$\overline{v(t)} = 0, \text{ (no switchings) } t \in [0, T] \tag{5.23}$$

b) If $\begin{cases} w_0 - \dfrac{k_2}{c_2} > 0 \\ \dfrac{c_1}{c_2} \cdot \dfrac{w_0 + q}{w_0 - \dfrac{k_2}{c_2}} < 1 - e^{-(w_0 + q)T} \end{cases}$ (5.24)

Then the optimal control has precisely 1 switching at $t = \theta$

$$\bar{v}(t) = \begin{cases} v_{0,} & t \in [0,\ \theta] \\ 0, & t \in (\theta,\ T] \end{cases} \tag{5.25}$$

The moment of switching $\theta$ is given by **(5.17)**.

Dr. Grigorieva solved **Problem 1** analytically and determined that in order to maximize the functional $J_1$, the optimal control $\bar{u}(t)$ must have at most one switching at $\tau \in (0, \theta)$ ($\theta$ is given by **(5.17)**)

$$\bar{u}(t) = \begin{cases} u_0, & t \in [0, \tau] \\ 0, & t \in [\tau, T] \end{cases} \tag{5.26}$$

The moment of switching $\tau$ can be found as:

$$\tau = \theta - \frac{1}{v_0 + q} \cdot \ln\left[1 + \frac{1 + \frac{k_1}{q}\left(1 - e^{-q(T-\theta)}\right)}{\frac{c_1 - 1}{v_0 + q}\left(v_0 - \frac{q + k_1}{c_1 - 1}\right)}\right] \tag{5.27}$$

This would allow the manufacturer to change production rate at most once during the planning period. Since we have a two criteria optimal control problem or cooperative game, when both **Problems 1** and **2** are solved, the optimal solution obtained in **Problem 2** will be substituted into the optimal solution for **Problem 1**. This will give the optimal solution of the **Main Problem**.

# CHAPTER VI

## COMPUTER MODELING

The statements proved in the previous chapter allow us to write a computer program that solves our problem numerically. We know that only controls $u(t)$ and $v(t)$ with at most one switching can lead to the optimal solution in **Problem 1** and **Problem 2,** respectively. Our computer program in MAPLE solves the Cauchy problem **(3.1)** for piecewise constant controls $u(t)$ and $v(t)$, where the moments of switching $\theta$ and $\tau$ for such controls change within the interval $[0, T]$. In this work we obtained numerical results for two cases:

a) when conditions **(5.22)** are true and $\bar{v}(t) = 0$

b) when conditions **(5.24)** are true, and $\bar{v}(t)$ is a piecewise constant function given by formula **(5.25)**.

Below we present graphs for situation b) obtained for different values of parameters $c_1, c_2, k_1, k_2, T, q, v_0, w_0,$ and $u_0$. Thus, let $0 \le u \le 300$, $q = 0.2$, $c_1$=\$3, $c_2$=\$10, $v_0$=1, $w_0$=1, $k_1 = k_2 = 1$, $T = 5$ days.

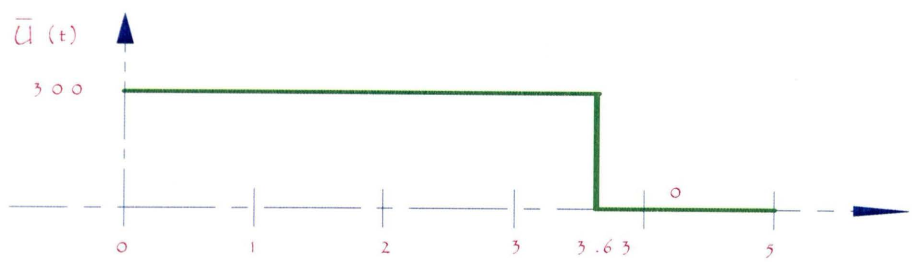


Figure 6.1 Manufacturer rate versus days

We found that the optimal control for a manufacturer is to change the rate of production at the end of the third day, and the concurrent optimal control for the retailer is to purchase goods until the middle of the fourth day, ($\boldsymbol{\theta} = \mathbf{4.57}$ days) , and then stop buying the goods. Since $\boldsymbol{w} = \boldsymbol{w}_0$ in $[\mathbf{0}, \boldsymbol{T}]$, the retailer continues to sell at the maximum rate. Maximum profit is \$700 for the manufacturer during the 5 day period.

The graph for the manufacturer's profit obtained as a function of two switchings (Fig 6.3) certainly indicates that the functional of profit depends on only one parameter (switching) as it was confirmed analytically.

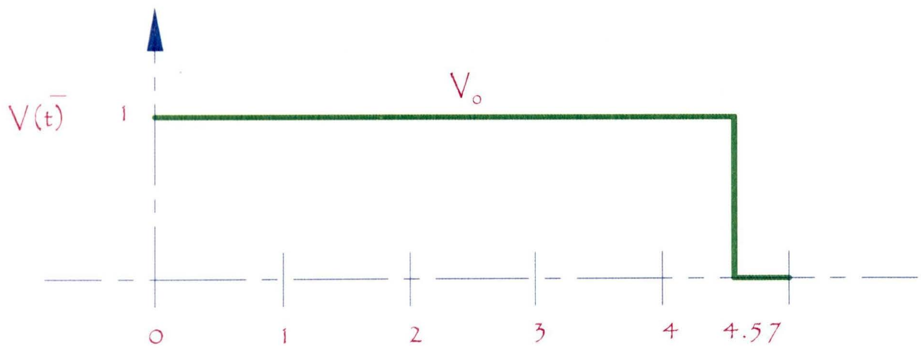


Figure 6.2 Retailer purchases versus days

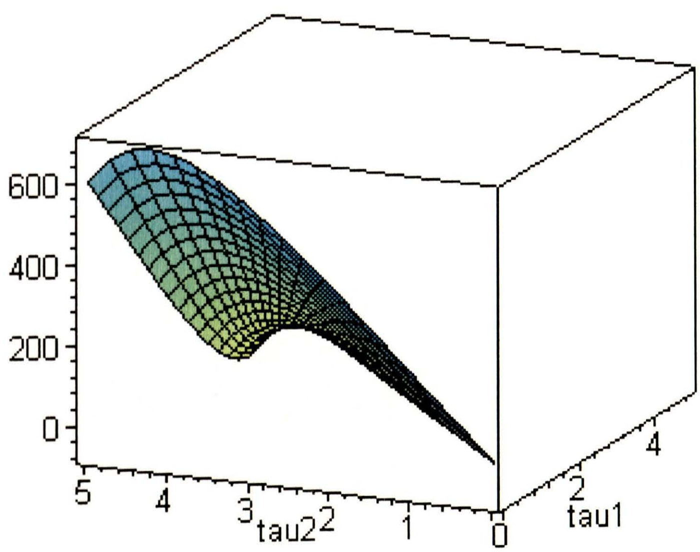


Figure 6.3 Functional of the manufacturer's profit versus tau1and tau2

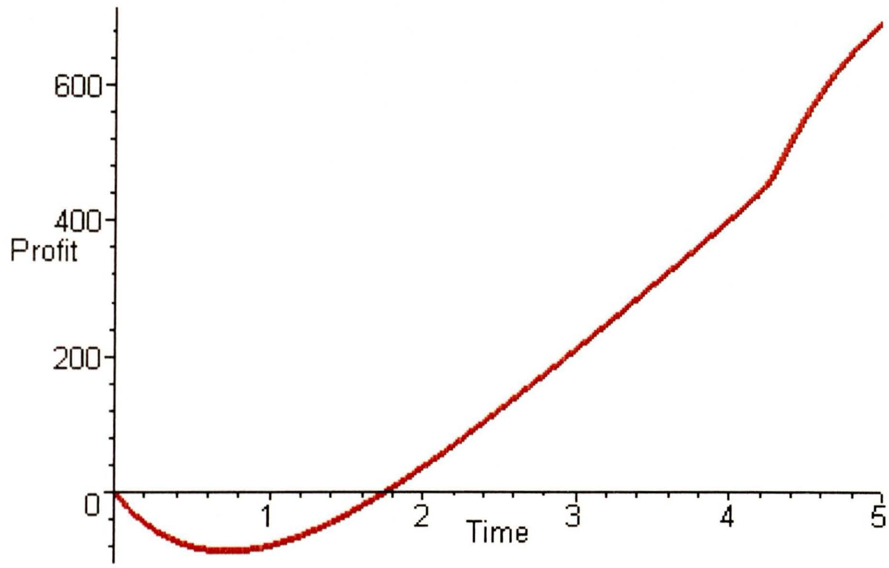


Figure 6.4 Manufacturer's profit versus time

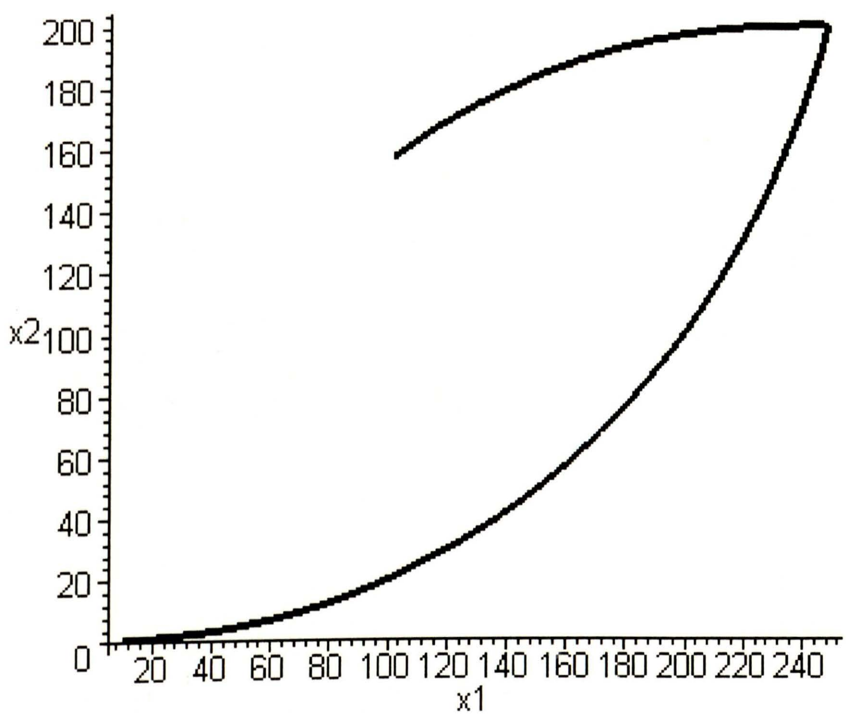


Figure 6.5 Phase portrait of system

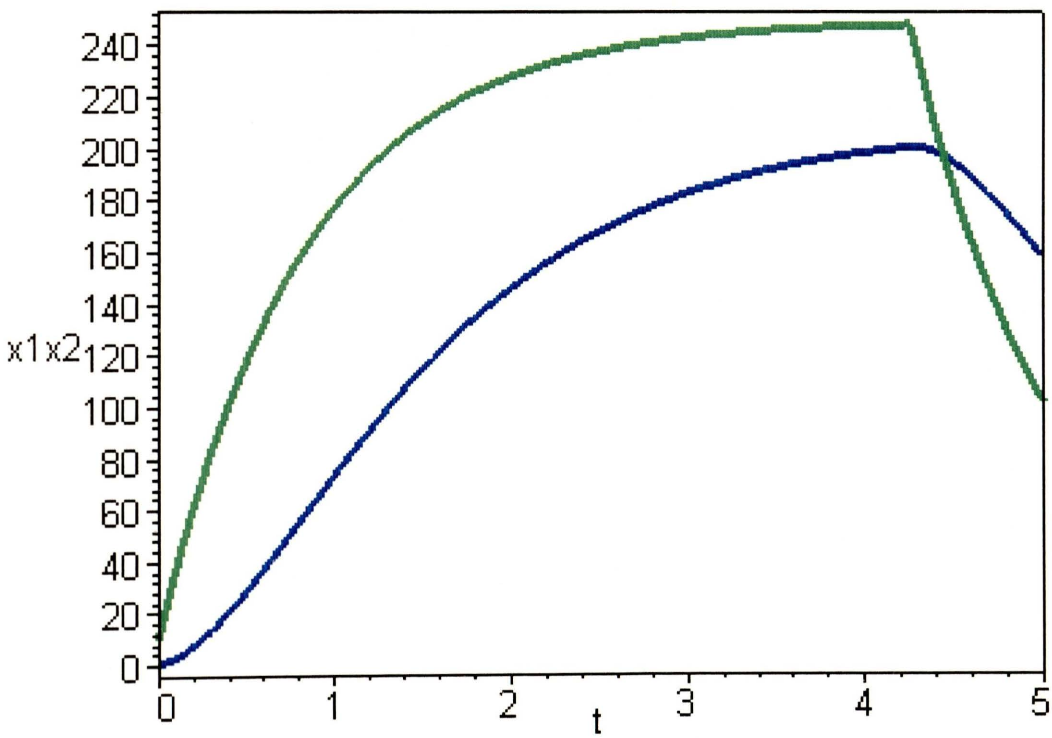


Figure 6.6 Optimal trajectories as functions of time

# CHAPTER VII

# CONCLUSIONS

Our results can be summarized as follows:

1. We modeled a cooperative game between a manufacturer and retailer by a bilinear system of differential equations.
2. We formulated a problem of profit maximization for both players using an optimal control problem and the Maximum Principle.
3. Using knowledge of optimal control theory, mathematical analysis, and differential equations, we obtained the solution to our problem analytically.
4. We investigated our control model for different values of the parameters.
5. Generalizing our results, the following scenario of the optimal players' activity can be proposed:

The manufacturer produces his consumer good during $[0,\tau]$ at the maximum production rate, then does not produce on $(\tau,T)$. The retailer starts buying the product at moment **t=0** and continues until $t=\theta$. We understand that on $(\tau,\theta)$, the manufacturer does not produce anymore, but the retailer continues buying the product. Thus, on $(\tau,T]$ neither production nor buying occurs. However, in order to maximize his profit the buyer (retailer) must sell his product at the constant rate, $w_0$, during the entire time interval **[0,T].**

Work on this project involved several things:

1. How to model a microeconomic process by a system of differential equations.
2. How to apply an optimal control problem and the Pontryagin Maximum Principal to solve a profit optimization problem.
3. How to use the Rolle's and Lagrange theorems to do proofs in advanced study.

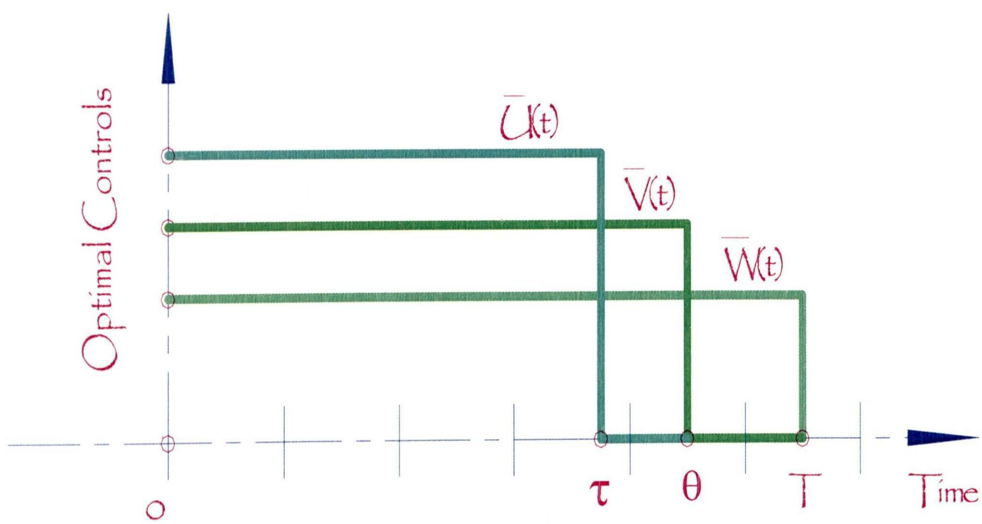


Figure 7.1 Optimal strategy for the manufacturer and the retailer

REFERENCES


1. Kirk, Donald. E., *Optimal Control Theory An Introduction* Englewood Cliffs, NJ: Prentice-Hall, Inc., 1970

2. Tu, Pierre N.V. *Introductory Optimization Dynamics,* 2nd ed. Berlin Heidelberg Springer-Verlag, 1991.

3. Shone, Ronald. *Economic Dynamics Phase Diagrams and their Economic Application* 2d ed. Cambridge, UK: Cambridge University Press, 2002

4. Davis, Morton D. *Game Theory A Nontechnical Introduction*, Books, Inc., New York, 1983. Dover Publications reprinted in 1997.

5. Grigorieva, Ellina V., Khailov, E. N., *On the attainability set for a nonlinear system in the plane,* Vestnik of Moscow University, **Computational Mathematics and Cybernetics**, N4 (2001), Translation by Alerton Press, Inc.,2002.

6. Grigorieva E. V.* and E.N. Khaïlov. "An Attainable Set of a Nonlinear Controlled Microeconomic Model'. **Journal of Nonlinear Dynamics and Control**, Vol. 11, No. 2, 157-176 (2005)

7. Paraev,Yu.I.,"A Game Approach to Production, Storage, and Marketing Problems'. **Automation and Remote Control,** Vol 66, Issue 2 , 272-280

8. Lee and Markus, *Foundations of Optimal Control Theory*; 1986, J. Wiley

9. Conrad, Bruce P., *Differential Equations with Boundary Value Problems* Upper Saddle River, N.J.: Prentice Hall, 2003.

10. Braun,M., *Differential Equations and Their Applications*, 3rd Edition, New York, N.Y., Springer-Verlag, 1983.

11. Johnson, R.M., *Linear and Differential Difference Equations,* West Sussex, England, Albion Publishing Limited, 1997.

12. Braun, Martin, *Differential Equation Models.* Volume 1, New York, N.Y., Springer-Verlag, 1983.

13. Marsden, J. and Weinstein, A., *Calculus I,* New York, N.Y., Springer-Verlag, 1985.