SUBLATTICES OF THE LATTICE OF TOPOLOGIES

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PREFACE

In comparison with other areas of mathematics, lattice theory is "new." Boole, in 1824, introduced a class of lattices (Boolean Algebras), and, in the 1890's, Schröder and Dedekind presented further concepts of lattice theory. In the 1930's the development of lattice theoretical concepts began to increase with publications by van der Waerden, von Neumann, Ore, Stone, Kantorovitch, Birkhoff, and others. Then in 1948, Garrett Birkhoff published his classical volume on lattice theory, in which he unified and up-dated former results and presented new discoveries [1,8].

Since 1950 R. W. Bagley [1], J. Hartmanis [5], and A. K. Steiner [7] have published articles specifically concerning the lattice of topologies. The purpose of this thesis is to examine the lattice of topologies and to determine whether certain subsets are sublattices.

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Dedication

Mr. and Mrs. Ernest Thompson

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CHAPTER I

GENERAL PROPERTIES OF THE LATTICE OF TOPOLOGIES

Presented in this chapter are basic definitions of lattice theory and general theorems concerning the lattice of topologies. A knowledge of fundamental topological properties and definitions is assumed. Unless otherwise indicated, the notation and basic definitions concerning lattices are those used by Thron [9] and/or Birkhoff [2]. All proofs are original, though some of the results presented in this paper are due to Steiner [7]. Some concepts, to my knowledge, have not been previously studied.

<u>Definition 1.1</u>: A <u>partially ordered set</u> is a pair (X, R) for which the relation R on X is reflexive, antisymmetric, and transitive.

<u>Definition 1.2</u>: An <u>upper bound</u> of a subset $A \subseteq X$, where X is a partially ordered set, is an element b of X such that for every $x \in A$, $x \leq b$.

<u>Definition 1.3</u>: A <u>least upper bound</u> of a subset $A \subseteq X$, where X is a partially ordered set, is an element b^* such that b^* is an upper bound for A and such that $b^* \leq b$ for $b \in B$, the set of all upper bounds of A.

<u>Note 1.1</u>: The least upper bound is also designated by the term supremum or sup and by the symbol $\bigvee$.

<u>Note 1.2</u>: Lower bound and greatest lower bound are similarly defined. The greatest lower bound is also designated by the term infimum or inf and by the symbol $\wedge$.

<u>Definition 1.4</u>: A <u>lattice</u> (L, R) is a partially ordered set (X, R) such that for any $a \in X$ and $b \in X$, $a \vee b$ exists and $a \wedge b$ exists.

<u>Theorem 1.1</u>: The collection of all topologies on a set X forms a lattice.

<u>Proof</u>: Let $\mathcal{T}$ be the set of all topologies on X. Consider the order relation as being set containment. Certainly for any $\mathcal{T}_\alpha \in \mathcal{T}$, $\mathcal{T}_\alpha \subseteq \mathcal{T}_\alpha$, so the relation is reflexive. Also if $\mathcal{T}_\alpha \subseteq \mathcal{T}_\beta$ and $\mathcal{T}_\beta \subseteq \mathcal{T}_\alpha$, then $\mathcal{T}_\alpha = \mathcal{T}_\beta$ by the property of set containment; thus, the relation is antisymmetric. Similarly, for topologies $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, $\mathcal{T}_\gamma$, if every open set in $\mathcal{T}_\alpha$ is in $\mathcal{T}_\beta$ and every open set in $\mathcal{T}_\beta$ is in $\mathcal{T}_\gamma$, then certainly every open set of $\mathcal{T}_\alpha$ is in $\mathcal{T}_\gamma$ and the relation is transitive.

Now consider $\mathcal{T}_\alpha \in \mathcal{T}$ and $\mathcal{T}_\beta \in \mathcal{T}$. To find $\mathcal{T}_\alpha \vee \mathcal{T}_\beta$, let $\mathcal{T}_\alpha \cup \mathcal{T}_\beta$ be a subbase. Then, by the definition of a subbase, taking finite intersections and arbitrary unions will generate a topology. Certainly it is the least such topology, since any upper bound must contain both $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$. Thus, $\mathcal{T}_\alpha \vee \mathcal{T}_\beta = \{\bigcup[\bigcap_{i=1}^{n}\{G_i : G_i \in \mathcal{T}_\alpha \cup \mathcal{T}_\beta\}]\}$.

To find the infimum, consider $\mathcal{T}_\alpha \cap \mathcal{T}_\beta$ (non-empty since the trivial topology is a subset of every topology). Since $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are topologies, certainly $\mathcal{T}_\alpha \cap \mathcal{T}_\beta$ is a topology. It is a lower bound since for every $G \in (\mathcal{T}_\alpha \cap \mathcal{T}_\beta)$, $G \in \mathcal{T}_\alpha$ and $G \in \mathcal{T}_\beta$; thus $(\mathcal{T}_\alpha \cap \mathcal{T}_\beta) \subseteq \mathcal{T}_\alpha$ and $(\mathcal{T}_\alpha \cap \mathcal{T}_\beta) \subseteq \mathcal{T}_\beta$. It is the greatest lower bound since any additional open sets could not belong to both $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$. Thus, $\mathcal{T}_\alpha \wedge \mathcal{T}_\beta = \mathcal{T}_\alpha \cap \mathcal{T}_\beta$.

<u>Definition 1.5</u>: A lattice (L, R) on a set X is <u>complete</u> if and only if for every $A \subseteq X$, sup A and inf A exist in X.

<u>Theorem 1.2</u>: The lattice of all topologies on a set X is complete.

<u>Proof</u>: Consider $\mathcal{T}$, the family of all topologies on X. For any subset $\{\mathcal{T}_\alpha\} \subseteq \mathcal{T}$, $\vee\{\mathcal{T}_\alpha\}$ is generated by using $\cup_\alpha\{\mathcal{T}_\alpha\}$ as a subbase. Similarly $\wedge\{\mathcal{T}_\alpha\} = \cap_\alpha\{\mathcal{T}_\alpha\}$.

<u>Definition 1.6</u>: A lattice (L, R) on X is <u>distributive</u> if and only if the operations $\vee$ and $\wedge$ satisfy the following property for every x, y, z $\in$ X:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

<u>Note 1.3</u>: Steiner [7] proved that the lattice of topologies is not distributive if X has three or more elements. The following example illustrates this for a set of three elements.

Let $X = \{a, b, c\}$. Consider the following topologies on X:

$\mathcal{T}_1 = \{\phi, X, \{b\}, \{a, c\}\}$;

$\mathcal{T}_2 = \{\phi, X, \{a\}, \{b, c\}\}$;

$\mathcal{T}_3 = \{\phi, X, \{b\}, \{a, b\}\}$.

Then $\mathcal{T}_1 \vee (\mathcal{T}_2 \wedge \mathcal{T}_3) =$
$\{\phi, X, \{b\}, \{a, c\}\} \vee$
$[\{\phi, X, \{a\}, \{b, c\}\} \wedge \{\phi, X, \{b\}, \{a, b\}\}] =$
$\{\phi, X, \{b\}, \{a, c\}\} \vee \{\phi, X\} = \{\phi, X, \{b\}, \{a, c\}\}$.
But $(\mathcal{T}_1 \vee \mathcal{T}_2) \wedge (\mathcal{T}_1 \vee \mathcal{T}_3) =$
$[\{\phi, X, \{b\}, \{a, c\}\} \vee \{\phi, X, \{a\}, \{b, c\}\}] \wedge$
$[\{\phi, X, \{b\}, \{a, c\}\} \vee \{\phi, X, \{b\}, \{a, b\}\}] =$
$[\{\phi, X, \{b\}, \{a, c\}, \{a\}, \{b, c\}, \{c\}, \{a, b\}] \wedge$
$[\{\phi, X, \{b\}, \{a, c\}, \{a, b\}, \{a\}\}] =$
$\{\phi, X, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$. So,
$\mathcal{T}_1 \vee (\mathcal{T}_2 \wedge \mathcal{T}_3) \neq (\mathcal{T}_1 \vee \mathcal{T}_2) \wedge (\mathcal{T}_1 \vee \mathcal{T}_3)$.

<u>Definition 1.7</u>: An element b of a partially ordered set $(X, <)$ is considered a <u>least</u> element of X if and only if for all $x \in X$, $b \leq x$.

<u>Note 1.4</u>: The largest element of X is similarly defined.

<u>Theorem 1.3</u>: The lattice of topologies on a set X has a largest and a least element, and hence is bounded.

Proof: Let $\{\mathcal{T}_\alpha\}$ be the set of all topologies on a set X. Since this set forms a complete lattice, the supremum and infimum exist for any subset of $\{\mathcal{T}_\alpha\}$. Certainly, $\bigcup_\alpha\{\mathcal{T}_\alpha\}$ is a subbase which generates the discrete topology. Also, $\bigcap_\alpha\{\mathcal{T}_\alpha\} = \{\phi, X\}$.

Definition 1.8: If a lattice has a least element b, and if there exists an element $c \neq b$ such that $b < x < c$ implies that $x = b$ or $x = c$, then c is an atom of the lattice.

Note 1.5: By Theorem 1.4, the least element in the lattice of topologies on any set X is the indiscrete topology; that is, $\mathcal{T}_b = \{\phi, X\}$. If $\mathcal{T}_c = \{\phi, X, G\}$ where $G \subseteq X$, then $\mathcal{T}_c$ is an atom, since for any $\mathcal{T}_x$ such that $\mathcal{T}_b \leq \mathcal{T}_x \leq \mathcal{T}_c$, either $\mathcal{T}_x = \mathcal{T}_b$ or $\mathcal{T}_x = \mathcal{T}_c$.

Definition 1.9: A base of a lattice L is a subset $B \subset L$ such that for every $x \in L$, other than the least element, there exists a set $B_x \subseteq B$ such that $x = \bigvee[b\colon\ b \in B_x]$.

Definition 1.10: An atomic lattice has a base consisting only of atoms.

Theorem 1.4: The lattice of topologies on any set X is an atomic lattice.

Proof: Let $\mathcal{T}$ be the set of all topologies on X. Let $\mathcal{B} = \{B_G\colon\ B_G = \{\phi, X, G\}::\ \ G \subseteq X\}$. Each B_G is an atom. Then for any $\mathcal{T}_x > \mathcal{T}_I$, where $\mathcal{T}_I = \{\phi, X\}$, there exists a subcollection from $\mathcal{B}$ such that for every $G \in \mathcal{T}_X$, G is in some B_G. Thus $\mathcal{T}_X = \bigvee\{B_G\}$.

Thus, the collection of topologies on any set X forms a lattice which is bounded, complete, non-distributive, and atomic.

CHAPTER II

SEPARATION PROPERTIES AND SUBLATTICES OF THE LATTICE OF TOPOLOGIES

After the identification of a lattice, it is natural to search for sublattices. The separation properties of T_0, T_1, T_2, R_0, R_1, and R_2 are apropriate to this investigation. The definitions of T_0-, T_1-, and T_2-spaces are those of Pervin [6] and of R_0-, R_1-, and R_2-spaces are those of Davis [4]. The closure of a set G is denoted $\overline{G}$, and the complement of G is denoted G'.

Definition 2.1: A subset L* of a lattice L is a sublattice if, for any two elements $x \in L^*$ and $y \in L^*$, $(x \vee y) \in L^*$ and $(x \wedge y) \in L^*$.

Note 2.1: It is possible for a subset of a lattice to be a lattice, but still not be a sublattice. However, it is always true that $x \vee^* y \geq x \vee y$ and that $x \wedge^* y \leq x \wedge y$, where $\vee^*$ and $\wedge^*$ are the infimum and supremum in the subcollection [9, p. 10].

Definition 2.2: A topological space X is a T_0-space if and only if for two distinct points $x \in X$ and $y \in X$, there exists an open set G which contains one of them but excludes the other.

Theorem 2.1: On a set X, where the cardinality of X is greater than or equal to two, the subset of the lattice of all T_0-topologies is not a sublattice.

Proof: Let $X = \{a, b\}$. Let $\mathcal{T}_1 = \{\phi, X, \{a\}\}$, and let $\mathcal{T}_2 = \{\phi, X, \{b\}\}$. Thus, $\mathcal{T}_1$ and $\mathcal{T}_2$ are T_0. However, $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{\phi, X\}$ which is not T_0.

Note 2.2: The example of Theorem 2.1 also shows that the subset consisting of T_0-topologies does not form a lattice.

Definition 2.3: A topological space X is a T_1-space if and only if for distinct points $x \in X$ and $y \in X$, there exist open sets G and H such that $x \in G$ with $y \notin G$ and $y \in H$ with $x \notin H$.

Note 2.3: Obviously, if a topological space is T_1, then it is also T_0.

Definition 2.4: A chain is a partially ordered set $(X, >)$ such that for every $x \in X$ and $y \in X$, either $x > y$, $y > x$, or $x = y$.

Definition 2.5: In a partially ordered set $(X, >)$, $m \in X$ is called a maximal element if and only if there exists no $x \in X$ with $x \neq m$ such that $x > m$.

Note 2.4: Zorn's Lemma states that if each chain in a partially ordered set has an upper bound, then there is a maximal element in the set.

Theorem 2.2: The subset consisting of all T_1-topologies on a set X is a lattice.

Proof: Let $\{\mathcal{T}_\alpha\}$ be the set of all topologies on a set X. Consider any two T_1-topologies $\mathcal{T}_i$ and $\mathcal{T}_j$. Then $\mathcal{T}_i \vee \mathcal{T}_j$ exists and is T_1 since whatever open sets existed in the topologies to insure T_1-ness also exist in the topology generated by the subbase $\mathcal{T}_i \cup \mathcal{T}_j$. (That is, topologies finer than a T_1-topology are T_1.)

Again consider T_1-topologies $\mathcal{T}_i$ and $\mathcal{T}_j$. Let

$A = \{\mathcal{T}_\alpha\colon \ \mathcal{T}_\alpha \subseteq \mathcal{T}_i, \mathcal{T}_\alpha \subseteq \mathcal{T}_j$, and $\mathcal{T}_\alpha$ is $T_1\}$;

that is, A is the set of all T_1-lower bounds for $\mathcal{T}_i$ and $\mathcal{T}_j$. Certainly $A \neq \phi$ since, for any set X, there exists a unique smallest T_1-topology (the cofinite topology).

Let $C = \{\mathcal{T}_n\}$ be a chain from A. Let $\bigcup\{\mathcal{T}_n\}$ be a subbase for a topology $\mathcal{T}$. Certainly $\mathcal{T}_n \subseteq \mathcal{T}$ for every n. Also $\bigcup\{\mathcal{T}_n\} \subseteq \mathcal{T}_i$ and $\bigcup\{\mathcal{T}_n\} \subseteq \mathcal{T}_j$. Therefore $\mathcal{T} \subseteq \mathcal{T}_i$, $\mathcal{T} \subseteq \mathcal{T}_j$, and $\mathcal{T}$ is T_1 since $\mathcal{T} = \bigvee\{\mathcal{T}_n\}$. Thus, $\mathcal{T} \in A$. By the construction of $\mathcal{T}$, it is an upper bound of the chain C in A. Thus, by Zorn's Lemma, there is a maximal element, say $\mathcal{T}_m$, in A. This implies $\mathcal{T}_i \wedge \mathcal{T}_j = \mathcal{T}_m$, a T_1-space.

Theorem 2.3: The lattice of all T_1-topologies is a sublattice of the lattice of all topologies on X.

Proof: Let $\mathcal{T}_i$ and $\mathcal{T}_j$ be T_1-topologies. Then $\mathcal{T}_i \vee \mathcal{T}_j$, the supremum of $\mathcal{T}_i$ and $\mathcal{T}_j$ over the lattice of all topologies, is T_1 (see Theorem 2.2).

Let $\mathcal{T}_i \wedge \mathcal{T}_j$ be the infimum of $\mathcal{T}_i$ and $\mathcal{T}_j$, over the lattice of all topologies, and let $\mathcal{T}_i \wedge^* \mathcal{T}_j$ be the infimum of $\mathcal{T}_i$ and $\mathcal{T}_j$ over the T_1-topologies. Then

$$\mathcal{T}_i \wedge^* \mathcal{T}_j \leq \mathcal{T}_i \wedge \mathcal{T}_j$$

by lattice theory. Since $\mathcal{T}_i \wedge^* \mathcal{T}_j$ is T_1 and topologies finer than a T_1-topology are also T_1, then $\mathcal{T}_i \wedge \mathcal{T}_j$ must be T_1; so $\mathcal{T}_i \wedge^* \mathcal{T}_j = \mathcal{T}_i \wedge \mathcal{T}_j$.

Definition 2.6: A topological space X is an R_0-space [4] if and only if either of the following is true:

(a) For all $x \in X$, $y \in X$, either $\overline{\{x\}} = \overline{\{y\}}$ or $\overline{\{x\}} \cap \overline{\{y\}} = \phi$;

(b) For every $x \in G \in \mathcal{T}$, $\overline{\{x\}} \subseteq G$.

Theorem 2.4: Parts (a) and (b) of Definition 2.6 are equivalent.

Proof: Suppose (a). Let $x \in G$. Suppose that $\overline{\{x\}} \not\subseteq G$. Then, $\overline{\{x\}} \cap G' \neq \phi$. Thus there exists an element y such that $y \in \overline{\{x\}}$ and $y \in G'$. Since $y \in \overline{\{x\}}$, then $\overline{\{y\}} \cap \overline{\{x\}} \neq \phi$, so $\overline{\{y\}} = \overline{\{x\}}$. But since $y \in G'$, $\overline{\{y\}} \subseteq G'$ since a closed set contains the closure of each of its points. Thus $\overline{\{x\}} \subseteq G'$ which is a contradiction since $x \in G$. Thus $\overline{\{x\}} \subseteq G$.

Suppose (b). Let $x \in G \in \mathcal{T}$. Thus $\overline{\{x\}} \subseteq G$. Let $y \in X$. Suppose $\overline{\{x\}} \neq \overline{\{y\}}$. There are two cases:

Case I. There exists an element z such that $z \in \overline{\{x\}}$ and $z \notin \overline{\{y\}}$. Since $z \notin \overline{\{y\}}$, then $z \in \overline{\{y\}}'$, an open set. This implies $x \in \overline{\{y\}}'$. So $\overline{\{x\}} \subseteq \overline{\{y\}}'$, and $\overline{\{x\}} \cap \overline{\{y\}} = \phi$.

Case II. There exists an element z such that $z \notin \overline{\{x\}}$ and $z \in \overline{\{y\}}$. Proof is similar to Case I.

<u>Theorem</u> <u>2.5</u>: The collection of all R_0-topologies on a space X forms a lattice.

<u>Proof</u>: (I) Let $\{\mathcal{T}_r\}$ be some set of R_0-topologies. Then, for every open set $G_i \in \mathcal{T}_{r_i}$, $x \in G_i$ implies that $\overline{\{x\}} \subseteq G_i$. $\operatorname{Sup}\{\mathcal{T}_r\}$ over the lattice of all topologies is generated by the subbase formed by $\bigcup_r\{\mathcal{T}_r\}$. Let $\sup\{\mathcal{T}_r\} = \mathcal{T}$. Certainly, $\overline{\{x\}}^{\mathcal{T}_r} \supseteq \overline{\{x\}}^{\mathcal{T}}$ for each r. Now, for any $G \in \mathcal{T}$, $G = \bigcup\{\bigcap_{i=1}^{n} G_i\}$, where $G_i \in \bigcup_r\{\mathcal{T}_r\}$. Certainly, $x \in G \in \mathcal{T}$ implies $x \in \left(\bigcap_{i=1}^{n} G_i\right)$ for some intersection, and since, for every G_i, $\overline{\{x\}}^{\mathcal{T}_{r_i}} \subseteq G_i$ (where $G_i \in \mathcal{T}_{r_i}$ for some r), then

$$\overline{\{x\}}^{\mathcal{T}} \subseteq \bigcap_{i=1}^{n} \overline{\{x\}}^{\mathcal{T}_{r_i}} \subseteq \bigcap_{i=1}^{n} G_i$$

and $\mathcal{T} = \sup\{\mathcal{T}_r\}$ is R_0.

(II) Similarly, let $\mathcal{T}_i$ and $\mathcal{T}_j$ be R_0-topologies on X. Let $\mathcal{A} = \{\mathcal{T}_r\colon\ \mathcal{T}_r \subseteq \mathcal{T}_i,\ \mathcal{T}_r \subseteq \mathcal{T}_j$, and $\mathcal{T}_r$ is $R_0\}$. Certainly $\mathcal{A} \neq \phi$ since the indiscrete topology is R_0 and is contained in $\mathcal{T}_i$ and $\mathcal{T}_j$. Let C be a chain from $\mathcal{A}$. By (I), sup C exists. Thus, by Zorn's Lemma, there is a maximal element, say $\mathcal{T}_m$, in $\mathcal{A}$, the set of R_0-lower bounds for $\mathcal{T}_i$ and $\mathcal{T}_j$, implying $\mathcal{T}_m = \mathcal{T}_i \wedge^* \mathcal{T}_j$, the infimum in the lattice of R_0-topologies.

Theorem 2.6: The intersection of two sublattices is a sublattice.

Proof: Let L be a lattice, L_1 and L_2 be sublattices, and $L_1 \cap L_2 = L^*$.

Case I. Suppose $L_1 \cap L_2 = L^* = \phi$. Since there are no nonempty subsets of ϕ, the supremum and infimum exist in L^*.

Case II. Suppose $L_1 \cap L_2 = L^* = \{a\}$. Then $\sup\{a\} = a$, $\inf\{a\} = a$, and $a \in L^*$.

Case III. Suppose $L_1 \cap L_2 = L^*$ where the cardinality of L^* is greater than or equal to two. Consider $A \in L^*$ and $B \in L^*$. Then certainly $A \in L_1 \subseteq L$ and $B \in L_1 \subseteq L$. Similarly, $A \in L_2 \subseteq L$ and $B \in L_2 \subseteq L$. Thus $A \vee^1 B$ exists in L_1, since L_1 is a lattice, and similarly $A \vee^2 B$ exists in L_2. Also since L_1 and L_2 are sublattices of L, $A \vee^1 B = A \vee^2 B = A \vee^L B$. Thus $A \vee^L B \in L_1 \cap L_2 = L^*$ and $A \vee^* B = A \vee^L B$.

Similarly, $A \wedge^* B \subseteq A \cap B \subseteq L_1$ and $A \wedge^* B \subseteq A \cap B \subseteq L_2$, since L_1 and L_2 are lattices. So $A \wedge^* B \subseteq A \cap B \subseteq L_1 \cap L_2 = L^*$. Since L_1, L_2 are sublattices, $A \wedge^1 B = A \wedge^2 B = A \wedge^L B$. So, $A \wedge^L B \in L_1 \cap L_2 = L^*$ and $A \wedge^* B = A \wedge^L B$.

Definition 2.7: A topology $\mathcal{T}$ is called a principal topology if and only if arbitrary intersections of open sets are open [7, p. 382].

Note 2.5: Every topology on a finite set X is principal.

Lemma 2.1: In a principal topology, the arbitrary union of closed sets is closed.

Proof: Let X be any space and $\mathcal{T}$ a principal topology on X. Let $K = \bigcup_\alpha \{F_\alpha\}$ where each F_α is a closed set. Then $K' = \bigcap_\alpha \{F_\alpha'\}$. Each F_α' is open; thus, by Definition 2.7, K' is open. Therefore $(K')' = K$ must be closed.

Theorem 2.7: A principal topology is R_0 if and only if every open set is closed.

Proof: Let $\mathcal{T}$ be a principal topology on a set X. Suppose every open set is also closed. Let $x \in G \in \mathcal{T}$. Thus, $\overline{\{x\}} \subseteq G$ since G is closed.

Now suppose $\mathcal{T}$ is an R_0-topology on X. Let $G \in \mathcal{T}$; thus, $\overline{\{x_i\}} \subseteq G$ for each $x_i \in G$. Taking the union of every point in G yields

$$G = \bigcup_i \{x_i\} \subseteq \bigcup_i \overline{\{x_i\}} = G.$$

By Lemma 2.1, $\bigcup_i \overline{\{x_i\}}$ is closed. Therefore G is closed.

Note 2.6: If every open set is closed, then certainly every closed set is also open.

Theorem 2.8: The principal R_0-topologies form a sublattice of the lattice of all topologies.

<u>Proof</u>: Let $\mathcal{T}_i$ and $\mathcal{T}_j$ be principal R_0-topologies on a set X. Part I of the proof of Theorem 2.4 shows that for any two R_0-topologies, their supremum is R_0; thus, the R_0-topologies form a sublattice with respect to the supremum. Steiner [7, p. 382] proved that the principal topologies form a sublattice. Thus, by Theorem 2.5, the set of topologies which are principal and R_0 will be a sublattice with respect to the supremum; that is, $\mathcal{T}_i \vee \mathcal{T}_j$ will be a principal R_0-topology.

Similarly, let $\mathcal{T}_i$ and $\mathcal{T}_j$ be principal R_0-topologies. Let $\mathcal{T}_i \wedge \mathcal{T}_j$ be their infimum over the lattice of all topologies. Certainly $\mathcal{T}_i \wedge \mathcal{T}_j$ is principal since the principal topologies are a sublattice of the lattice of topologies. Suppose $G \in \mathcal{T}_i \wedge \mathcal{T}_j = \mathcal{T}_i \cap \mathcal{T}_j$. Then $G \in \mathcal{T}_i$ and $G \in \mathcal{T}_j$. Since $\mathcal{T}_i$ and $\mathcal{T}_j$ are principal R_0-topologies, $G' \in \mathcal{T}_i$ and $G' \in \mathcal{T}_j$, by Theorem 2.6. Therefore, again by Theorem 2.6, $\mathcal{T}_i \wedge \mathcal{T}_j$ is R_0. Therefore, since $\mathcal{T}_i \wedge \mathcal{T}_j$ is principal and R_0, the collection of principal R_0-topologies is a sublattice.

<u>Note 2.7</u>: Whether or not the lattice of all R_0-topologies forms a sublattice of the lattice of all topologies over an infinite set is not known.

<u>Theorem 2.9</u>: A topology is T_1 if and only if it is both R_0 and T_0.

Proof: Suppose $\mathcal{T}$ is a T_1-topology on a set X. Thus, $\mathcal{T}$ is also T_0. Also for every $\{x\}$, $\overline{\{x\}} = \{x\}$. Thus, if $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}} = \phi$, and $\mathcal{T}$ is R_0.

Suppose $\mathcal{T}$ is a topology which is both T_0 and R_0. If $x \neq y$, $\overline{\{x\}} \neq \overline{\{y\}}$, since $\mathcal{T}$ is T_0. So $\overline{\{x\}} \cap \overline{\{y\}} = \phi$, since $\mathcal{T}$ is R_0. Thus $y \in \overline{\{x\}}' \in \mathcal{T}$, and $x \in \overline{\{y\}}' \in \mathcal{T}$. Therefore $\mathcal{T}$ is T_1.

Definition 2.8: A topological space $(X, \mathcal{T})$ is an R_1-space [4] if and only if $\overline{\{x\}} \neq \overline{\{y\}}$ implies there exist open sets $G_x \in \mathcal{T}$ and $G_y \in \mathcal{T}$ such that $\overline{\{x\}} \subseteq G_x$ and $\overline{\{y\}} \subseteq G_y$ and such that $G_x \cap G_y = \phi$.

Theorem 2.10: If $(X, \mathcal{T})$ is R_1, then it is R_0.

Proof: If $(X, \mathcal{T})$ is an R_1-topological space in which for all points x and y elements of X, $\overline{\{x\}} = \overline{\{y\}}$, then $\mathcal{T}$ is R_0. Consider $(X, \mathcal{T})$, an R_1-space and points x and y such that $\overline{\{x\}} \neq \overline{\{y\}}$. Thus, there exist $G_x \subseteq \mathcal{T}$ and $G_y \subseteq \mathcal{T}$ such that $\overline{\{x\}} \subseteq G_x$ and $\overline{\{y\}} \subseteq G_y$ and $G_x \cap G_y = \phi$. Therefore, $\overline{\{x\}} \cap \overline{\{y\}} = \phi$ and $\mathcal{T}$ is R_0.

Theorem 2.11: For any principal topology $\mathcal{T}$ on a set X, $\mathcal{T}$ is R_1 if and only if $\mathcal{T}$ is R_0.

Proof: Suppose $\mathcal{T}$ is R_1; then by Theorem 2.9, it is R_0.

Suppose $\mathcal{T}$ is R_0. There are two cases to consider:

Case I: For every $x, y \in X$, $\overline{\{x\}} = \overline{\{y\}}$. Then $\mathcal{T}$ is R_1 vacuously.

Case II. Suppose $\overline{\{x\}} \neq \overline{\{y\}}$. Therefore, since $\mathcal{T}$ is R_0, $\overline{\{x\}} \cap \overline{\{y\}} = \phi$. By Note 2.6, $\overline{\{x\}}$ and $\overline{\{y\}}$ are open as well as closed. Thus $\overline{\{x\}}$ and $\overline{\{y\}}$ are contained in open sets (respectively $\overline{\{x\}}$ and $\overline{\{y\}}$) which are disjoint.

<u>Theorem</u> <u>2.12</u>: The principal R_1-topologies on a set X form a sublattice of the lattice of topologies.

<u>Proof</u>: See Theorems 2.8 and 2.11.

<u>Note</u> <u>2.8</u>: A topological space which is R_1 may or may not also be T_1 and/or T_0. For example, consider the following topologies on the set $X = \{a, b, c\}$:

$\mathcal{T}_1 = \{\phi, X, \{a\}, \{a, c\}\}$;

$\mathcal{T}_2 = \{\phi, X, \{b\}, \{a, c\}\}$;

$\mathcal{T}_3 = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Of these, $\mathcal{T}_1$ is T_0 but not R_1; $\mathcal{T}_2$ is R_1 but not T_1; and $\mathcal{T}_3$ is both T_1 and R_1.

<u>Definition</u> <u>2.9</u>: A topological space $(X, \mathcal{T})$ is a <u>T_2-space</u> (or Hausdorff space) if and only if, for distinct points x and y in X, there exist disjoint open sets, one containing x and the other containing y.

<u>Note</u> <u>2.9</u>: A T_2-topological space is automatically T_1.

Theorem 2.13: A topological space $(X, \mathcal{T})$ is T_2 if and only if it is T_1 and R_1.

Proof: Suppose $(X, \mathcal{T})$ is T_1 and R_1. Let x and y be distinct points in X. Then $\overline{\{x\}} \neq \overline{\{y\}}$ since singletons are closed in T_1-topologies. By the property of being R_1, there exist G_x and G_y such that $\overline{\{x\}} \subseteq G_x$ and $\overline{\{y\}} \subseteq G_y$ and $G_x \cap G_y = \phi$. Thus $\mathcal{T}$ is T_2.

Now suppose $\mathcal{T}$ is T_2. Let x and y be distinct points in X. Then there exist open sets G_x and G_y such that $x \in G_x$ and $y \in G_y$ and $G_x \cap G_y = \phi$. By being T_2, $\mathcal{T}$ is also T_1, which implies that singletons are closed. Therefore, $\{x\} = \overline{\{x\}} \subseteq G_x$, $\{y\} = \overline{\{y\}} \subseteq G_y$, $G_x \cap G_y = \phi$, and $\mathcal{T}$ is R_0.

Theorem 2.14: The collection of all T_2-topologies on an arbitrary set X does not form a lattice (and hence is not a sublattice).

Proof: Let X equal the set of real numbers. Consider the following topologies on X:

$$\mathcal{T}_1 = \{G: \ 0 \notin G \text{ or } 0 \in G \text{ and } G' \text{ is finite}\};$$

$$\mathcal{T}_2 = \{H: \ 1 \notin H \text{ or } 1 \in H \text{ and } H' \text{ is finite}\}.$$

By Pervin [6, p. 79], each of these topologies (known as Fort's space) is T_2. Then $\mathcal{T}_1 \wedge \mathcal{T}_2 =$ $\mathcal{T}_1 \cap \mathcal{T}_2 = \{K: \ 0 \notin K, 1 \notin K, \text{ or } 0 \text{ or } 1 \in K \text{ and } K' \text{ is finite}\}$. Consider the points 0 and 1 and open sets K_0 and K_1 such that $0 \in K_0 \in \mathcal{T}_1 \cap \mathcal{T}_2$ and $1 \in K_1 \in \mathcal{T}_1 \cap \mathcal{T}_2$. Since their complements are finite, K_0 and K_1 must be infinite. However, $K_0 \cap K_1 \neq \phi$ since this would imply that one of them would be contained in the complement of the other (that is, an infinite set contained in a finite set). Therefore, there do not exist disjoint open sets, one containing x and the other y; thus $\mathcal{T}_1 \wedge \mathcal{T}_2$ is not Hausdorff.

Theorem 2.15: The set of all R_1-topologies does not form a sublattice of the lattice of topologies.

Proof: Let $\{\mathcal{T}_r\}$ be the set of R_1-topologies and $\{\mathcal{T}_i\}$ be the set of T_1-topologies on a set X. Then the set of topologies in the intersection of $\{\mathcal{T}_r\}$ and $\{\mathcal{T}_i\}$ is composed of T_2-topologies by Theorem 2.13. Suppose $\{\mathcal{T}_r\}$ is a sublattice; by Theorem 2.3, $\{\mathcal{T}_i\}$ is a sublattice. Therefore $\{\mathcal{T}_r\} \cap \{\mathcal{T}_i\}$ is a sublattice by Theorem 2.6. But this contradicts Theorem 2.14; therefore, $\{\mathcal{T}_r\}$ must not be a sublattice.

Note 2.10: The counterexample given in Theorem 2.14 also illustrates that the infimum of two R_1-topologies need not be R_1.

Definition 2.9: A topological space $(X, \mathcal{T})$ is an R_2-space if and only if it satisfies the following: If F is a closed subset of X and $x \in X$, $x \notin F$, then there exist open sets G_F and G_x such that $F \subseteq G_F$, $x \in G_x$, and $G_F \cap G_x = \phi$.

Note 2.11: An R_2-space is sometimes referred to as a regular space. Also note that since the trivial topology is R_2(vacuously), an R_2-topological space need not be T_2, T_1, or even T_0.

Theorem 2.16: If $(X, \mathcal{T})$ is R_2, then it is R_1 (and hence R_0).

Proof: Let $(X, \mathcal{T})$ be R_2. Let $x, y \in X$. If $\overline{\{x\}} = \overline{\{y\}}$, then $\mathcal{T}$ is R_1. Suppose $\overline{\{x\}} \neq \overline{\{y\}}$. Then $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$; else $\overline{\{x\}} \subseteq \overline{\{y\}}$ and $\overline{\{y\}} \subseteq \overline{\{x\}}$. Since $\mathcal{T}$ is R_2, there exist open sets G_x such that $x \in G_x$ and G_y such that $\overline{\{y\}} \subseteq G_y$, and $G_x \cap G_y = \phi$.

Suppose $\overline{\{x\}} \not\subseteq G_x$. Then $\overline{\{x\}} \cap (G_x)' \neq \phi$. So there exists an element z such that $z \in (\overline{\{x\}} \cap (G_x)')$. Thus $z \in (G_x)'$, a closed set, but $x \notin (G_x)'$. So there exist open sets U and V such that $(G_x)' \subseteq U$, $x \in V$, and $U \cap V = \phi$. But this implies $U\backslash\{z\} \cap \{x\} = \phi$, so $z \notin \overline{\{x\}}$ for any $z \in (G_x)'$. Thus $\overline{\{x\}} \subseteq G_x$. Since $G_y \cap G_x = \phi$, $\mathcal{T}$ is R_1.

Theorem 2.17: In a principal topology $\mathcal{T}$ on X, $\mathcal{T}$ is R_2 if and only if $\mathcal{T}$ is R_1.

Proof: Suppose $\mathcal{T}$ is R_2. Then by Theorem 2.16, it is R_1.

Suppose $\mathcal{T}$ is R_1. By Theorem 2.10, it is also R_0. Let F be a closed subset of X and $x \in X$, $x \notin F$.

For any $y \in F$, $\overline{\{y\}} \subseteq F$, so $\overline{\{y\}} \neq \overline{\{x\}}$. Thus there exist open sets G_x and G_y such that $\overline{\{x\}} \subseteq G_x$ and $\overline{\{y\}} \subseteq G_y$. Since $\mathcal{T}$ is principal and R_0, every open set is also closed by Theorem 2.7. Therefore y is an element of a closed set, G_y, which is contained in an open set, G_y; also $x \in G_x$ and $G_y \cap G_x = \phi$. Thus $\mathcal{T}$ is R_2.

Theorem 2.18: The principal R_2-topologies form a sublattice of the lattice of topologies.

Proof: See Theorems 2.8 and 2.12.

Theorem 2.19: The subset of all R_2-topologies on an infinite set does not form a sublattice.

Proof: Let X equal the set of real numbers. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be defined as in Theorem 2.14. Each of these topologies is R_2, as can be seen by considering either of them. Therefore, consider

$$\mathcal{T}_1 = \{G: 0 \notin G \text{ or } 0 \in G \text{ and } G' \text{ is finite}\}.$$

Let F be a closed subset of X. Let $x \in X$ such that $x \notin F$. There are three cases to consider.

Case I. $x = 0$. Then for any open set G which contains 0, G' is finite and closed since G is open. However, G' is also open since $0 \notin G'$. So there exist disjoint open sets G' and G such that G' (closed) is contained in G' (open) and $0 \in G$.

Case II. $x \neq 0$, $0 \notin F$. Since F is closed and $0 \notin F$, F' is open and contains 0. As in Case I, since $0 \notin F$, F is an open set. So there exist an open set F which contains F and an open set F' such that $x \in F'$, and $F \cap F' = \phi$.

Case III. $x \neq 0$, $0 \in F$. Since $x \neq 0$, $\{x\}$ is an open set. So $\{x\}'$ is a closed set. But since $0 \in \{x\}'$ and $(\{x\}')'$ is finite, $\{x\}'$ is open as well as closed. Therefore, there exist open sets $\{x\}$ and $\{x\}'$ such that $x \in \{x\}$ and $F \subseteq \{x\}'$ and $\{x\} \cap \{x\}' = \phi$.

Now consider $\mathcal{T}_1 \wedge \mathcal{T}_2 =$ $\mathcal{T}_1 \cap \mathcal{T}_2 = \{K: 0 \notin K$ and $1 \notin K$, or 0 or $1 \in K$ and K' is finite$\}$. Consider the singleton set $\{0\}$. This is a closed set since $K = \{0\}'$ is such that $0 \notin K$, $1 \in K$, and K' is finite. Certainly, there exists an open set K^* such that $1 \in K^*$, $0 \notin K^*$ and $(K^*)'$ is finite. However for any open set K^0 such that $\{0\} \subseteq K^0$, $(K^0)'$ is finite; therefore, $K^* \cap K^0 \neq \phi$ since this would imply that one of them would be contained in the complement of the other. Therefore, $\mathcal{T}_1 \wedge \mathcal{T}_2$ is not R_2.

Definition 2.10: A topological space $(X, \mathcal{T})$ is a T_3-space if and only if it is R_2 and T_1.

Theorem 2.20: The principal T_3-topologies form a sublattice of the lattice of topologies.

Proof: See Theorems 2.3, 2.6, and 2.18.

In summary, the collections of T_0- and T_2-topologies on a set are not lattices (and hence not sublattices), while the set of T_1-topologies is a sublattice of the lattice of topologies. The set of R_0-topologies form a lattice, and the set of principal R_0-topologies (hence all R_0-topologies on a finite set) is a sublattice. The question of whether all R_0-topologies form a sublattice is not answered. Though principal R_1- and principal R_2-topologies are sublattices of the lattice of topologies, neither the sets of R_1- nor R_2-topologies are lattices in general.

CHAPTER III

MISCELLANEOUS TOPOLOGICAL PROPERTIES AND SUBLATTICES

While the separation axioms help to identify topological spaces, topologies are further characterized as being connected, compact, Lindelöf, scattered, or metrizable. In the lattice of topologies, the subsets composed of connected topologies, compact topologies, Lindelöf topologies, and scattered topologies fail to be lattices because the supremum may fail to be connected, compact, or Lindelöf, respectively. The subset of metrizable topologies is a sublattice with respect to the supremum; however, it is not known if the infimum of two metrizable topologies is metrizable.

The notation and definitions are those of Pervin [6].

Definition 3.1: If given a topological space $(X, \mathcal{T})$ and sets A, B, and E contained in X, then sets A and B form a separation of E if and only if $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$, $A \cap d(B) = \phi$, $B \cap d(A) = \phi$, and $A \cup B = E$.

Definition 3.2: A set is connected if and only if it has no separation.

Theorem 3.1: In the lattice of all topologies on a set X, the subset consisting of all topologies which are connected does not form a sublattice.

Proof: Let X = {a, b, c}. Let

$$\mathcal{T}_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\} \text{ and}$$

$$\mathcal{T}_2 = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}.$$

Thus, $\mathcal{T}_1$ and $\mathcal{T}_2$ are connected; however, $\mathcal{T}_1 \vee \mathcal{T}_2$ equals the discrete topology which is not connected.

Definition 3.3: If $\{A_\lambda\}$ is a family of sets and $S \subseteq \bigcup_\lambda A_\lambda$, then $\{A_\lambda\}$ is called a covering of S. If each A_λ is open, then $\{A_\lambda\}$ is called an open covering of S.

Definition 3.4: Given $(X, \mathcal{T})$ and $S \subseteq X$, S is compact if and only if every open covering of S contains a finite subcovering.

Theorem 3.2: In the lattice of all topologies on an uncountable set X, the subset consisting of all compact topological spaces does not form a sublattice.

Proof: Let X = Reals. Let

$$\mathcal{T}_1 = \{\phi, X, \{G_a\colon\ G_a = [a, \infty)\}\} \text{ and}$$

$$\mathcal{T}_2 = \{\phi, X, \{H_b\colon\ H_b = (-\infty, b]\}\}.$$

Consider [m, n] a closed interval of X. Certainly [m, n] is compact with $\mathcal{T}_1$, since for any open cover of [m, n], say $\mathcal{O} = \{[a_i, \infty)\colon\ i \in I\}$, there exists a finite subcover, which is the single open set $[a_i, \infty)$ where $a_i = m$. Similarly, $[m, n]$ is compact with $\mathcal{T}_2$.

The supremum of $\mathcal{T}_1$ and $\mathcal{T}_2$ is formed by the subbase $(\mathcal{T}_1 \cup \mathcal{T}_2)$. There exists an open cover consisting of singleton sets for which there does not exist a finite subcover. Thus $\mathcal{T}_1 \vee \mathcal{T}_2$ is not compact.

Definition 3.5: A topological space $(X, \mathcal{T})$ is a Lindelöf space if and only if every open covering is reducible to a countable subcovering.

Theorem 3.3: In the lattice of all topologies on an uncountable set X, the subset of Lindelöf topologies does not form a sublattice.

Proof: See the proof of Theorem 3.2.

Definition 3.6: A subset E of a topological space $(X, \mathcal{T})$ is dense-in-itself if and only if every point of E is a limit point of E; that is, if $E \subseteq d(E)$.

Definition 3.7: The nucleus of a set E is the union of all dense-in-itself subsets of E.

Definition 3.8: A set whose nucleus is empty is called scattered.

Theorem 3.4: In the lattice of topologies on any set X, the subset of scattered topologies does not form a sublattice.

Proof: Let $X = \{0, 1\}$. Consider the following topologies on X:

$$\mathcal{T}_1 = \{\phi, X, \{0\}\} \text{ and}$$

$$\mathcal{T}_2 = \{\phi, X, \{1\}\}.$$

Therefore, $(X, \mathcal{T}_1)$ is scattered since $E \not\subseteq d(E)$ for any nonempty $E \subseteq X$, which implies that there exist no nonempty dense-in-itself sets and the nucleus of X is empty. Similarly, $\mathcal{T}_2$ is scattered. However, $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{\phi, X\}$, the trivial topology which is not scattered since for the subset X, $X \subseteq d(X) = \{0, 1\}$, and the nucleus of X is not empty.

Definition 3.9: A metric for a set X is a mapping d of $X \times X$ into the non-negative reals satisfying, for all $x, y, z \in X$, the following axioms:

(a) $d(x, x) = 0$;

(b) $d(x, y) \leq d(x, y) + d(y, z)$;

(c) $d(x, y) = d(y, x)$;

(d) if $x \neq y$, then $d(x, y) > 0$.

Definition 3.10: If $x \in X$ with metric d and ε is any positive real number, then the set of all $y \in X$, where $d(x, y) < \varepsilon$ will be called the ball with center at x and radius ε. This ball is denoted $B(x, \varepsilon)$.

Note 3.1: By Pervin [6, p. 100], the collection of all balls of points on a set X induces a topology for X.

<u>Definition 3.11</u>: A topological space $(X, \mathcal{T})$ is <u>metrizable</u> if and only if there exists a metric for X which induces $\mathcal{T}$.

<u>Note 3.2</u>: Not all topologies are induced by metrics, and those topologies that are metrizable may be induced by many metrics, but a single metric induces a unique topology.

<u>Theorem 3.5</u>: In the lattice of topologies on a set X, the supremum of any subset consisting of metrizable topologies is metrizable.

<u>Proof</u>: Consider $\mathcal{T}_d$ and $\mathcal{T}_p$, metrizable topologies on X. Thus there exists a metric d which induces $\mathcal{T}_d$ and a metric p which induces $\mathcal{T}_p$. Then $\mathcal{T} = \mathcal{T}_d \vee \mathcal{T}_p$ is formed by the subbase $\mathcal{T}_d \cup \mathcal{T}_p$. Define a metric q such that

$$q(x, y) = \max(d(x, y), p(x, y))$$

for all $x, y \in X$.

This is a metric since it satisfies the axioms of Definition 3.6:

(a) $q(x, x) = \max(d(x, x), p(x, x)) = \max(0, 0) = 0$.

(b) Consider the following:

$$d(x, y) \leq \max(d(x, y), p(x, y));$$
$$p(x, y) \leq \max(d(x, y), p(x, y));$$
$$d(y, z) \leq \max(d(y, z), p(y, z));$$
$$p(y, z) \leq \max(d(y, z), p(y, z)).$$

Certainly, $d(x, z) \leq d(x, y) + d(y, z) \leq$
$\max(d(x, y), p(x, y)) + \max(d(y, z), p(y, z))$,
and $p(x, z) \leq p(x, y) + p(y, z) \leq$
$\max(d(x, y), p(x, y)) + \max(d(y, z), p(y, z))$.
So, $\max(d(x, z), p(x, z)) \leq$
$\max(d(x, y), p(x, y)) + \max(d(y, z), p(y, z))$ which, by definition of q, is $q(x, z) \leq q(x, y) + q(y, z)$.

(c) $q(x, y) = \max(d(x, y), p(x, y)) =$
$\max(d(y, x), p(y, x)) = q(y, x)$.

(d) If $x \neq y$, then
$q(x, y) = \max(d(x, y), p(x, y)) > 0$, since both $d(x, y) > 0$ and $p(x, y) > 0$.

Thus q is a metric. Now it is necessary to show that $\mathcal{T}_d \vee \mathcal{T}_p$ is the same topology on X as that generated by the metric q.

Let $\mathcal{T} = \mathcal{T}_d \vee \mathcal{T}_p = \{\cup[\bigcap_{i=1}^{n}\{G_i : G_i \in (\mathcal{T}_d \cup \mathcal{T}_p)\}]\}$.

Let $\mathcal{T}_q$ be the topology induced by the metric q where $q(x, y) = \max(d(x, y), p(x, y))$.

(I) Choose $G \in \mathcal{T}$, and let $x \in G$. So $x \in \bigcap_{i=1}^{n}\{G_i\} \subseteq G$, for some intersection. For every i such that $x \in G_i$, there exists an ε_i such that

$B_d(x, \varepsilon_i) \subseteq G_i$ for $G_i \in \mathcal{T}_d$, or
$B_p(x, \varepsilon_i) \subseteq G_i$ for $G_i \in \mathcal{T}_p$.

Let $\varepsilon = \min\{\varepsilon_i\}$. If $G_i \in \mathcal{T}_d$, then

$$B_q(x, \varepsilon) \subseteq B_d(x, \varepsilon) \subseteq B_d(x, \varepsilon_i) \subseteq G_i.$$

Similarly, if $G_i \in \mathcal{T}_p$, then

$$B_q(x, \varepsilon) \subseteq B_p(x, \varepsilon) \subseteq B_p(x, \varepsilon_i) \subseteq G_i.$$

Therefore, $B_q(x, \varepsilon) \subseteq G_i$, for every i. So, $B_q(x, \varepsilon)$ is contained in the intersection of all G_i which contain x. This implies $B_q(x, \varepsilon) \subseteq \bigcap_{i=1}^{n}\{G_i: \ x \in G_i\} \subseteq G$. Thus,

$$\mathcal{T} \subseteq \mathcal{T}_q.$$

(II) Now, choose $G^* \in \mathcal{T}_q$ and let $x \in G^*$. There exists a $B_q(x, \varepsilon) \subseteq G^*$. Consider

$$B_d(x, \varepsilon) \cap B_p(x, \varepsilon) \in \mathcal{T}_d \vee \mathcal{T}_p = \mathcal{T}.$$

Let $y \in B_d(x, \varepsilon) \cap B_p(x, \varepsilon)$. So $d(x, y) < \varepsilon$, and $p(x, y) < \varepsilon$. Since $q(x, y) = \max(d(x, y), p(x, y))$, $q(x, y) < \varepsilon$. So $y \in B_q(x, \varepsilon)$, and

$$B_d(x, \varepsilon) \cap B_p(x, \varepsilon) \subseteq B_q(x, \varepsilon) \subseteq G^* \in \mathcal{T}_q.$$

Thus,

$$\mathcal{T}_q \subseteq \mathcal{T}.$$

Combining the results of (I) and (II), $\mathcal{T} = \mathcal{T}_q$; therefore, the supremum of any two metrizable topologies is metrizable and the topology induced by the metric is equivalent to the sup topology.

<u>Note</u> <u>3.3</u>: Consider the following construction: For topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that Axioms (a), (b), and (c) are satisfied, define a function f such that

$$f(x, y) = \min(d(x, y), p(x, y)).$$

Then let

$$f^*(x, y) = \inf\{\sum_{i=1}^{n} f(x_i, x_{i+1}): x_0 = x \text{ and } x_{n+1} = y\}.$$

By Birkhoff [3], the function f* is the infimum of the "distance functions" d and p. However, f* does not satisfy Axiom (d), so f*, though "close" to being a metric which would induce the infimum of any two metrizable topologies, is not sufficient. Whether such a metric does exist is not known.

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