AN INVESTIGATION OF SYSTEMS OF POPULATION MODELS

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DEDICATION

I dedicate this thesis to my dear mother, Nell Nichols, who has spent endless hours sweeping up dog hair and keeping my house clean while I completed my Master's Degree.

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ABSTRACT

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The purpose of this thesis is to possibly help nontraditional, first- generation math students in their attempt at learning ordinary differential equations systems of population models with positive coefficients. I hope to accomplish this by building a guide containing some of the prerequisite mathematical concepts, demonstrating the procedures used while investigating the previously mentioned types of systems. The definitions of the vocabulary that is used to fully understand this process, the use of parameter population system to demonstrate the graphing program PPLANES, the explanation of the graphs created while using this program, and the use of Maple (Version 10) to find eigenvectors of the equilibrium points four case scenarios.

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CHAPTER I

INTRODUCTION

Being a nontraditional, first-generation college student lends special insight into the abyss other nontraditional students experience during their endeavor to complete a bachelor’s degree in mathematics. Some of the primary problems nontraditional students experience include the following: the financial strains, taking more years to complete their degree than other students who are second (or more) generation students, not retaining all the mathematical concepts from previous classes that force students to relearn the prerequisite mathematical concepts while taking new classes, and working 60 to 90 hour weeks while taking their college courses.

Nontraditional math students may have trouble juggling all their responsibilities, along with finding time to seek help, when they do not comprehend their mathematical assignments. The resources available to college math students vary from institution to institution. Most institutions offer an on- campus math lab for their students, but nontraditional students may have issues meeting with a tutor due to their work schedules and family responsibilities. Ohio State University (OSU), for example, provides tutors, workshops, and online math resources (https://mslc.osu.edu/about/location). These resources include practice exams and their solutions, and interactive online lessons for a few of the OSU mathematical courses (https://mslc.osu.edu/math-1151-online-lessons) for students having trouble. Students that attend other institutions may also use Ohio State University’s website to help build their math skills.

Two other examples of online math aids for lower-level math students are Khan Academy (2006) and Leap of Faith Financial Services Inc, which is the *The World of Math Online* (2000–2005). Both of these online sites target elementary and high school students, but college students may find these sites useful to understand basic concepts of their mathematical courses. Students who use applications (app) on their cellphone may wish to download and use Photomath (2017). This app allows students to scan math questions that are handwritten, typed in a textbook, typed in an online math program, or typed in directly into the app's calculator. The app then displays the solution to these questions. Students may also choose to view step-by-step instructions on how to solve the math questions.

Postsecondary institutions seem to be unaware of the problems that nontraditional, first-generation math students face. Consequently, this problem has not been fully addressed by postsecondary institutions; therefore, the purpose of this thesis was to possibly help nontraditional, first-generation undergraduate math students in their attempt at learning ordinary differential equation systems of population models with positive coefficients. This was accomplished by building a guide containing some of the prerequisite mathematical concepts and demonstrating the procedures used while solving the aforementioned types of population systems, creating a detailed look at the graphing program PPlanes, and defining the terminology that may be used during this process.

What is a nontraditional, first-generation college student? This has not been defined as a grouped term, but the characteristics of a nontraditional college student have been defined in the *Findings from the Condition of Education 2002: Nontraditional Undergraduates* (Choy, 2002). Students are considered "highly nontraditional" if they have four or more of the characteristics, are "moderately nontraditional" if they have two or three of these traits, and are "minimally nontraditional" if they have only one of the following characteristics traits: A student who delays enrollment (age of 24 or older), a student who attends part-time for at least a portion of the academic year, a student who works full-time (35 hours or more per week) while enrolled, a student who is considered financially independent on eligibility for financially aid, a student who has dependents other than a spouse, a student who is a single parent, and/or a student who does not have a high school diploma (Choy, 2002).

A first-generation college student has no clear definition (Smith, 2015). Some of the significant characteristics of a first-generation student are listed in the U.S. News & World Report as the following: A first-generation college student usually delays starting college (around the age of 22) and has parents with no college experience or parents with no bachelor's degree. These students will start at a non-four-year institution and will take six years to complete their bachelor's degree (Boyington, 2015). Why is there a need to help these students? Nationally, only 50% of the first-generation college students who enrolled at a 4-year university in 2004 earned their degrees within 6 years (DeAngelo, Franke, Hurtado, Pryor, & Tran, 2011).

The Texas Higher Education Coordinating Board's (2016) new proposal, 60x30TX, targeting 60 percent of Texans, between the age of 25–34, will have a certificate or college degree by the year 2030. Texans, who are 25 years and older, who have some college credits but no degree number 3.8 million in 2014. Also, 4-year students who dropped out (stopped out) of higher education with 90 or more semester credit hours (Bachelor of Science in Mathematics requires 120 credit hours) numbered 48,000 between the years 2008 and 2012. During this same time, at two-year colleges, the number of students with 55 or more semester credit hours (60 hours required Associate of Science degree) who dropped out (stopped out) was 161,000 (Texas Higher Education Coordinating Board, 2016).

Clearly, the guide will not help all these students but will hopefully provide help to returning students to recall forgotten mathematical concepts and help them complete their degrees.

CHAPTER II

PREREQUISITE MATHEMATICS

Basic Algebra

In mathematics, a binomial is a two-term polynomial [e.g., $(x + 3)$], then we can also consider the product of two binomials is as follows, for example: $(x + 3)(x - 5)$. When multiplying two binomials, there are numerous methods used. The most popular methods are the distributive property and the FOIL (First, Outer, Inner, Last) method. The distributive property method distributes the second binomial to each term of the first binomial and then any like terms are combined if necessary:

$(x + 3)(x - 5)$

$= (x - 5) + 3(x - 5)$

$= (x)(x) + (x)(-5) + (3)(x) + (3)(-5) =$

$x^2 - 5x + 3x - 15.$

Combining these like terms results in the trinomial: $x^2 - 2x - 15$. The FOIL method multiplies each term in the parenthesis in this order: the first two terms, the outer two terms, the inner two terms, and the last two terms; then combine like terms if necessary. A modified method of the FOIL is demonstrated. First, draw two goal posts (purple and red lines) under the two binomials $(x + 3\ )(\ x - 5\ )$.

Second, multiply the inner two terms (what the purple line touches, positive three times x) and place the product under the purple line (the signed term: $+\,3x$). Then multiply the outer two terms (what the red line touches, x times negative five) and place the product above the red line (the signed term: $-\,5x$) $(x + 3\ )(\ x - 5\ )$.

$+3x$

$-5x$

Third, when possible, combine like terms and then place under the red line; however, but if no like terms exist, then write these two terms under the red line including their signs:

$(x + 3\ )(\ -5\ ).$

$+3x$

$-5x$

Fourth, place a capital "F" in front of the red line, and place a capital "L" on the back of the red goal post as shown below:

$(x + 3\ )(\ -5\ ).$

$+3x$

$F \quad -5x \quad L$

$-2x$

The "F" is the product of the first terms in each parenthesis as shown: $F = (x)(x) = x^2$ then place this term under the red line in front of the combined like term. The "L" is the product of the last terms in each parenthesis as follows: $L = (3)(-5) = -15$. Then place this term under the red line behind the combined like term as follows:

$(x + 3\ )(\ -5\ ).$

$+3x$

$F \quad -5x \quad L$

$x^2 \quad -2x \quad 15$

This completes the process of multiplying a set of binomials using a modified method of the FOIL.

A quadratic expression, $ax^2 + bx + c$, may be factored (UNFOILED), which is basically the process shown above in reverse. A few more steps are added to verify the solution. Start with the quadratic expression: $x^2 - 5x - 6$. Then first step to this process is to look for the greatest common factor (GCF). A GCF is the greatest factor that is common to each term.

Second, if GCF is the factor of one, then draw a large goal post (blue line) from the first term to the last term. Third, place the factors of the first term on the left-hand side of the goal post positioned under the first term and the factors of the last term located under the right-hand side of the goal post as shown below:

$x^2 - 5x - 6.$

$(x)(x) \qquad (1)(6)$

$(2)(3)$

When all the factors are listed, write two sets of parentheses with purple and red lines (goal posts). The first position of each set of parentheses is where one of the factors of the first term of the trinomial is placed. The last position of each set of parentheses is where one of the factors of the third term of the trinomial is placed. We will look at the case where the leading coefficient of the squared term is one; it then becomes a question of which of the two factors, when multiplied, will be the last term negative six; however, when added will be the coefficient of the middle term negative five. The fourth step is to place a set (first and third term) of factors into the parentheses and then multiply to check the solution. The factors of x^2 and 6 are placed inside of the parentheses and factored using a modified method of the FOIL:

$x^2 - 5x - 6.$

$(x \quad 1)(x \quad 6)$

$1x$

$6x$

Choose the signs that when combined must be negative five from the original trinomial. Using -6 and $+1$ results in -5. The terms (purple and red) are combined and then placed under the red line as shown:

$(x \quad 1)(x \quad 6)$

$1x$

$6x$

$5x$

Determining how to place the two signs correctly in the parentheses is how most students make a mistake. Using this method, the first sign (purple number) is placed in the first parentheses, and the second sign (red number) is placed in the second parentheses. The "F" terms are multiplied and placed under the red line, then the "L" terms are multiplied and placed under red line (as the last term), shown here:

$x^2 - 5x - 6.$

$(x + 1)(- 6)$

$1x$

$F \quad 6x \quad L$

$x^2 - 5x - 6.$

A quadratic expression is factored correctly when the new trinomial (under red line) is the original trinomial. Then the solution is the product of the two parentheses. Students need to pay close attention to the sign between the second and third term of the resulting trinomial. When the wrong set of factors is used, then the sign between the last two terms will be different from the original quadratic expression. If these signs are wrong, the factors used were the wrong choice. Switch the placement of the same factors and try again using this same process. When the signs are again wrong, use the other set of factors until the trinomial under the red line matches the original trinomial.

The original quadratic expression set to zero will create a quadratic equation in the standard form: $x^2 - 5x - 6x = 0$. Then the last step of solving for a quadratic equation is to solve for the possible real roots by setting the terms in each pair of parentheses equal to zero and then solve for the variable. The quadratic equation is then in standard form and is solved for the possible real roots as follows:

$x^2 - 5x - 6x = 0$

$(x + 1)(x - 6) = 0.$

The parentheses are set to zero and then solve as shown: $x + 1 = 0$. Then subtract negative one from both sides as shown: $x = -1$. The second real root is as follows: $x - 6 = 0$. Then add six to both sides, resulting as follows: $x = 6$. Then possible solutions of this equation are as shown: $x = -1, 6$. Each of these solutions must be checked for erroneous (meaning invalid) answers by substituting one solution at a time into the original equation, and then simplified. When both of the statements are true after simplifying, then the possible real solutions are the real roots of the equation.

Two other popular methods used to solve quadratic equations are completing the square and the quadratic formula. The concept of completing the square is to make a perfect square trinomial that when factored will reduce into a quantity being squared. The formulas for a perfect square trinomial are shown here: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$. When solving for a quadratic equation using completing the square method, first start with the equation in standard form: $x^2 + 8x + 7 = 0$. The coefficient of the squared term (first term) must be a positive one; if not, divide each term by the coefficient.

Second, move the third term to the right-hand side of the equation. Then take the middle term, divide it by two, and square the results. Third, add this new term to both sides of the equation to make a perfect square trinomial on the left-hand side of the equation:

$x^2 + 8x = -7$

$x^2 + 8x + 16 = -7 + 16$.

Combine the like terms on the right-hand side of the equation: $x^2 + 8x + 16 = 9$. The left-hand side of the equation is a perfect square trinomial and will simplify to the quantity squared: $(x + 4)^2 = 9$. The fourth step is to use the square root property to reduce the squared term to a single quantity. The square root property is a way to solve for the real roots of a quadratic equation, but there are a few rules that must be followed. The numerical coefficient must be a positive one, and the squared term must be isolated, preferably on the left-hand side of the equation. Also, remember that a square root negative number is complex. The equation has the squared term isolated on the left-hand side. The solution using the square root property results as follows:

$(x+4)^2 = 9$

$\sqrt{(x+4)^2} = \pm\sqrt{9}$

$|x+4| = \pm\sqrt{9}$

$x+4 = \pm 3.$

The fifth step is solving for the two real roots by solving for the variable using the addition property of equality. This means to isolate x we add the additive inverse (a term with opposite sign) to both sides of the equation. The results are as follows:

$x + 4 - 4 = -4 + 3$

$x_1 = -1.$

The second solution was as follows:

$x_2 + 4 - 4 = -4 - 3$

$x_2 = -7.$

The two possible real solutions are as shown: $x_1 = -1, -7$. The solutions will need to be checked for erroneous answers. This process is started with checking the solution of negative one, as shown:

$(-1)^2 + 8(-1) + 7 = 0$

$1 - 8 + 7 = 0.$

The statement is true; therefore, -1 is a real solution of this equation. Finish by checking for the possible solution of -7:

$(-7)^2 + 8(-7) + 7 = 0$

$49 - 56 + 7 = 0.$

This statement is also true; therefore, -7 is also a real solution for this equation.

This completes solving for real roots by using completing the square; using the quadratic formula is less complicated since there are less rules to remember. Using the quadratic formula is an easier method than completing the square to solve for quadratic equations. The philosophy of Rosalie Reiter was, "I show the hardest methods first in class, so my students will appreciate the easier methods." Even YouTube has several videos to help students learn the formula, including the one sung to the "Pop Goes the Weasel" song. Start with the quadratic equation being in standard form: $ax^2 + bx + c = 0$. This should be completed before using the quadratic formula and the letters *a*, *b*, and *c* represent real numbers. The quadratic formula is as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is read as the variable *x* equals the opposite of *b* plus/minus the square root of the quantity of b squared minus four times *a* times *c* all over two times *a*. Using the quadratic equation from completing the square, the two possible solutions are as follows:

$$x^2 + 8x + 7 = 0$$

$$x_{1,2} = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(7)}}{2(1)}$$

$$x_{1,2} = \frac{-8 \pm \sqrt{64 - 28}}{2}$$

$$x_{1,2} = \frac{-8 \pm \sqrt{36}}{2}$$

$$x_{1,2} = \frac{-8 \pm 6}{2}.$$

Solving for first possible real root is shown as follows:

$$x_1 = \frac{-8 + 6}{2} = \frac{-2}{2} = -1.$$

Now, solve for the second possible root, shown here:

$$x_2 = \frac{-8 - 6}{2} = \frac{-14}{2} = -7.$$

The solutions were checked during the completing the square process; there is no need to repeat this step. The quadratic formula is also used to find the nature of the roots of a quadratic equation. The discriminant of a quadratic formula is located under the radical: $b^2 - 4ac$.

The discriminant is used to discover the nature of the possible solutions of a quadratic equation. There are four types of possible solutions. A quick review of all four examples follows. The first equation, $6x^2 + 7x + 2 = 0$, will have a positive perfect square as the discriminant and will have two different rational (no radical or imaginary numbers in the answer) solutions. Starting with the equation, substitute the numerical coefficients including their signs (if negative numbers) into the discriminant and simplify as follows:

$6x^2 + 7x + 2 = 0$

$b^2 - 4ac = 7^2 - (4)(6)(2) = 49 - 48 = 1.$

One is a positive perfect square; there will be two different rational solutions. The second equation, $3x^2 + 4x - 2 = 0$, will have a positive discriminant, but not a perfect square. Start with the equation; then substitute the numerical coefficients into the discriminant as follows:

$3x^2 + 4x - 2 = 0$

$4^2 - 4(3)(-2) = 16 + 24 = 40.$

This equation will have two different irrational solutions (answer will contain radicals).

The third equation, $4x^2 - 12x + 9 = 0$, will have a discriminant of zero and will have only one rational solution: $4x^2 - 12x + 9 = 0$

$(-12)^2 - (4)(4)(9) = 144 - 144 = 0.$

The final equation, $x^2 + 4 = 0$, results in two complex conjugates. This case the discriminant is negative, which means that the roots of this equation are complex and contains an imaginary number denoted by i.

The equation $x^2 + 4 = 0$ is chosen with $b = 0$. This type of equation seems to confuse students, since the coefficient b equals zero. The method to solve for this type quadratic equation is shown as follows:

$$x_{1,2} = \frac{-0 \pm \sqrt{0^2 - (4)(1)(4)}}{2(1)} = \frac{\pm\sqrt{-16}}{2}.$$

Notice the discriminant is located under the radical and is negative. The radical may be simplified by substituting $i^2 = -1$, as follows:

$$x_{1,2} = \frac{\pm\sqrt{16(-1)}}{2} = \frac{\pm i\sqrt{16}}{2} = \pm\frac{4i}{2} = \pm 2i.$$

This equation has two complex, not real, answers and will contain the i notation. Knowing how to find the discriminant is a useful tool when graphing quadratic functions: $(x) = x^2 + 3x - 4$. This concludes three methods to solve a quadrative equation.

A linear system that contains two equations with two variables is the last concept of this section. Seeing how or if these two lines cross each other is the main focus of this demonstration. There are several different ways to solve for a linear system. This section will concentrate on solving a linear system by using the substitution method and then explain three types of solutions students may encounter.

A system of two linear equations with two variables is a prime candidate to solve by the substitution method if one of these two equations has (or by solving will have) a variable with a coefficient of negative or positive one. When solving for this type of system, only one variable at a time will be found. Then the first solution will be evaluated in the remaining equation to find the second variable. Start with the linear system, shown below:

$$\begin{cases} 14x - 2y = 12 \\ \ \ 3x + 2y = 22 \end{cases}.$$

The first equation will have a coefficient $x = 7$. Using the first equation in the system above, we will solve for y in terms of x as follows:

$14x - 2y = 12$
$-2y = 12 - 14x$
$y = -6 + 7x.$

Why this method is called substitution is because the answer from this equation will be substituted for the y–variable in the second equation as shown:

$3x + 2(y) = 22$
$3x + 2(-6 + 7x) = 22.$

Notice that there is now only the x–variable in this equation. Simplify the equation and solve for the x–variable:

$3x - 12 + 14x = 22$
$17x - 12 = 22$
$17x = 34$
$x = 2.$

This solution will now be substituted into the first equation for the x–variable to find the solution for the y–variable as shown:

$y = -6 + 7(x)$
$y = -6 + 7(2)$
$y = -6 + 14$
$y = 8.$

This system of linear equations has one unique solution at the point $(2, 8)$. This means that when these two lines are graphed they will intersect (cross) only once, and it will be at the point $(2, 8)$. These equations are independent and are said to be a consistent system of equations, since there exists a point that is consistent with both equations. Some linear systems are said to be inconsistent when the two equations are parallel lines. This means when these lines are graphed they will never intersect. An example of an inconsistent system is shown here:

$$\begin{cases} 4x + 2y = 10 \\ 2x + y = 3 \end{cases}.$$

Using the substitution method, start by solving for the y–variable in the second equation:

$2x + y = 3$

$y = 3 - 2x.$

Substitute this solution in the first equation for the y–variable and simplify:

$4x + 2(y) = 10$

$4x + 2(3 - 2x) = 10$

$4x + 6 - 4x = 10.$

Combine like terms on the left-hand side of the equation: $6 = 10$. Oops! Where did the x–variable go? Six does not equal ten: $6 \neq 10$. This system of linear equations is considered inconsistent and will never intersect, for they are parallel lines. This system is said to have no solution. Some systems of linear equations are said to have an infinite number of solutions. These types of systems are said to have dependent equations, meaning the same line will be graphed, one over the other. An example of a system of linear systems with dependent equations is shown below:

$$\begin{cases} 2x + \ y = \ 6 \\ 4x + 2y = 12 \end{cases}.$$

The y–variable in the first equation is solved, and then substitute the solution into the second equation as follows:

$2x + y = 6$

$y = 6 - 2x$

$4x + 2(y) = 12$

$4x + 2(6 - 2x) = 12$

$4x + 12 - 4x = 12.$

The last step is to reduce the left-hand side of the equation to 12. This makes a true statement of $12 = 12$, and there is an infinite number of solutions since the two equations in this system represent the same line. This concludes the basic algebra section.

Matrices

A matrix is made of horizontal rows and vertical columns that form a rectangular array of elements. The orders of the elements in matrices are first listed by rows, and then by columns. Matrix $\boldsymbol{A}_{2\,x\,2}$ (matrix is in boldface print) is said to have two horizontal rows and two vertical columns with the elements listed below:

$$\boldsymbol{A}_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}.$$

A transpose of a matrix is accomplished by rewriting the rows of matrix $\boldsymbol{A}$ to the columns of the transpose matrix $\boldsymbol{A}_{\mathrm{T}}$, read $\boldsymbol{A}$–Transpose, as shown:

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}.$$

By assigning integers for the elements in matrix $\boldsymbol{A}$:

$$\boldsymbol{A}_{2x2} = \begin{bmatrix} 2 & 9 \\ -5 & -1 \end{bmatrix}.$$

The trace of this matrix, denoted by Tr$\boldsymbol{A}$ or tr$\boldsymbol{A}$, is the sum of the diagonal elements:

$$tr\boldsymbol{A} = \begin{vmatrix} 2 & 9 \\ -5 & -1 \end{vmatrix} = 2 + (-1) = 1.$$

The determinant of matrix $\boldsymbol{A}$ (denoted by det($\boldsymbol{A}$) or $|\boldsymbol{A}|$) is the first row-first column element, multiplied by the second row-second column element, and subtracted by the product of first row-second column element, multiplied by the second row-first element. This is very confusing in words, so an example of determinant of matrix $\boldsymbol{A}$ is shown:

$$|\boldsymbol{A}| = \begin{vmatrix} 2 & 9 \\ -5 & -1 \end{vmatrix} = (2)(-1) - (9)(-5) = -2 + 45 = 43.$$

The addition of two matrices is accomplished by adding each element in the first matrix by each of the corresponding elements in the second matrix. This means the matrices must be of the same order (size), or they cannot be added. Given matrix $\boldsymbol{A}$ and matrix $\boldsymbol{B}$, an example of adding these two matrices of the same order is as shown:

$\boldsymbol{A}_{2x2} = \begin{bmatrix} 2 & 9 \\ -5 & -1 \end{bmatrix}$, and $\boldsymbol{B}_{2x2} = \begin{bmatrix} 10 & -6 \\ -2 & 15 \end{bmatrix}$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 2 & 9 \\ -5 & -1 \end{bmatrix} + \begin{bmatrix} 10 & -6 \\ -2 & 15 \end{bmatrix} = \begin{bmatrix} 2+10 & 9+(-6) \\ -5+(-2) & -1+15 \end{bmatrix} = \begin{bmatrix} 12 & 3 \\ -7 & 14 \end{bmatrix}.$$

A scalar multiple of a matrix is computed by multiplying each element in a matrix by the scalar. This scalar can be either a real or complex number. The scalar may be represented by the Greek letter lambda λ. An identity matrix is a matrix of any size that has the diagonal elements of one, and the remaining elements are zero. The identity matrix multiplied by the scalar λ is as follows:

$$\lambda \boldsymbol{I}_{2x2} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

An example of a scalar of 3 multiplying the given matrix A is shown below:

$$3\boldsymbol{A} = 3\begin{bmatrix} 2 & 9 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} 3*2 & 3*9 \\ 3*(-5) & 3*(-1) \end{bmatrix} = \begin{bmatrix} 6 & 27 \\ -15 & -3 \end{bmatrix}.$$

When subtracting two matrices $\boldsymbol{A} - \boldsymbol{B}$, it is less confusing if matrix $\boldsymbol{B}$ is first multiplied by scalar of negative one, and then these two matrices are added. This process is shown as follows:

$$-\boldsymbol{B}_{2x2} = -1 \begin{bmatrix} 10 & -6 \\ -2 & 15 \end{bmatrix} = \begin{bmatrix} -10 & 6 \\ 2 & -15 \end{bmatrix}$$

$$\boldsymbol{A} + (-\boldsymbol{B}) = \begin{bmatrix} 2 & 9 \\ -5 & -1 \end{bmatrix} + \begin{bmatrix} -10 & 6 \\ 2 & -15 \end{bmatrix} = \begin{bmatrix} 2+(-10) & 9+6 \\ -5+2 & -1+(-15) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 15 \\ -3 & -16 \end{bmatrix}.$$

When multiplying two matrices, special rules apply. A major difference is the elements of the first matrix are not multiplied by the corresponding elements of the second matrix, as in addition. Instead, matrix $\boldsymbol{C}_{2x3}$ may multiply matrix $\boldsymbol{D}_{3x1}$ if and only if the columns of matrix $\boldsymbol{C}_{2x3}$ equals the same number of rows in matrix $\boldsymbol{D}_{3x1}$. Checking the size of these two matrices, $(2\ x\ 3)(3\ x\ 1)$, the two inner numbers are the same; therefore, these two matrices may be multiplied. A fun fact is that the size of the product matrix of these two matrices will be the outer numbers $(2\ x\ 1)$. The multiplication process of matrix $\boldsymbol{C}_{2x3}$ and matrix $\boldsymbol{D}_{3x1}$ is to first multiply the first element in row one of $\boldsymbol{C}_{2x3}$ with the first element in column $\boldsymbol{D}_{3x1}$ added to the multiplied second element in row one of $\boldsymbol{C}_{2x3}$ with the second element in column $\boldsymbol{D}_{3x1}$. Then add the multiplied third element in row one of $\boldsymbol{C}_{2x3}$ with the third element in column $\boldsymbol{D}_{3x1}$ [e.g., $(2)(5) + (-1)(6) + (7)(7)$]. This will complete the first row of the product matrix.

The second row of the product matrix is the same process listed above, but the elements of the second row of $\boldsymbol{C}_{2x3}$ will multiply to the elements in column of matrix $\boldsymbol{D}_{3x1}$ [e.g., $(5)(5) + (0)(6) + (-3)(7)$], as shown below:

$$\boldsymbol{C}_{2x3} = \begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & -3 \end{bmatrix}$$

$$\boldsymbol{D}_{3x1} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

$$\boldsymbol{C}_{2x3}\boldsymbol{D}_{3x1} = \begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

$$\boldsymbol{CD} = \begin{bmatrix} (2)(5) + (-1)(6) + (7)(7) \\ (5)(5) + (0)(6) + (-3)(7) \end{bmatrix}$$

$$\boldsymbol{CD} = \begin{bmatrix} 10 + (-6) + 49 \\ 25 + 0 + (-21) \end{bmatrix}$$

$$\boldsymbol{CD} = \begin{bmatrix} 53 \\ 4 \end{bmatrix}.$$

While first learning the concept of multiplying two matrices, it is perhaps helpful to draw a horizontal arrow over matrix $\boldsymbol{C}$ (→) and a vertical arrow (↓) over matrix $\boldsymbol{D}$ to lessen the confusion.

A review of the matrix operations may help while solving systems of simultaneous linear equations (equations that share at least one solution point for two unknown variables usually x and y) by rewriting the systems in matrix equation form of $\boldsymbol{Ax} = \boldsymbol{b}$ (vectors $\boldsymbol{x}$ and $\boldsymbol{b}$ will be in bold lowercase letters) and then solving for the unknown variables. The $\boldsymbol{A}$ is the coefficient matrix (coefficients of all unknown variables) of $\boldsymbol{x}$ (all the variables), and $\boldsymbol{b}$ is the solution for the system. A system of linear equations may be solved for the unknown variables written in form, $\boldsymbol{Ax} = \boldsymbol{b}$ by using the previous linear system and rewriting this system in the form of a matrix equation, shown below:

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{cases} 14x - 2y = 12 \\ 3x + 2y = 22 \end{cases}$$

$$\boldsymbol{A} = \begin{bmatrix} 14 & -2 \\ 3 & 2 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 12 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} 14 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 22 \end{bmatrix}.$$

This matrix may be solved by Gaussian elimination. First, write the matrix as an augmented matrix, which is designated by the symbol $\boldsymbol{A}^{\mathbf{b}}$ (Bronson & Costa, 2009). The Gaussian elimination method solves linear equations by using elementary row operations that includes the interchanging of two rows, multiplying a row by a non-zero scalar, and adding one row to another row. The line between the coefficient matrix $\boldsymbol{A}$ and the column matrix $\boldsymbol{b}$ is used to indicate the equals sign between these two matrices.

When considering the augmented matrix, let R_1 be defined as row one, and R_2 be defined as row two. The augmented matrix with the elementary row reductions is as follows:

$$\boldsymbol{A^b} = \left[\begin{array}{cc|c} 14 & -2 & 12 \\ 3 & 2 & 22 \end{array}\right]$$

$$\left\{R_1 \to \frac{1}{14}R_1 \qquad \boldsymbol{A^b} = \left[\begin{array}{cc|c} 1 & -\frac{1}{7} & \frac{6}{7} \\ 3 & 2 & 22 \end{array}\right]\right.$$

$$\{R_2 \to R_2 - 3R_1 \qquad \boldsymbol{A^b} = \left[\begin{array}{cc|c} 1 & -\frac{1}{7} & \frac{6}{7} \\ 0 & \frac{17}{7} & \frac{136}{7} \end{array}\right]$$

$$\left\{R_2 \to \frac{7}{17}R_2. \qquad \boldsymbol{A^b} = \left[\begin{array}{cc|c} 1 & -\frac{1}{7} & \frac{6}{7} \\ 0 & 1 & 8 \end{array}\right].\right.$$

The augmented matrix is then rewritten into a system of linear equations:

$$\begin{cases} x - \dfrac{1}{7}y = \dfrac{6}{7}. \\ y = 8 \end{cases}$$

The solution $y = 8$ is back-substituting into the first equation. Solve for the $x-$ variable as follows:

$$x - \frac{1}{7}(8) = \frac{6}{7}$$

$$x - \frac{8}{7} = \frac{6}{7}$$

$$x = \frac{8}{7} + \frac{6}{7}$$

$$x = \frac{14}{7} = 2.$$

This verifies the solution set from the previous section of the simultaneous linear system's solution of the point $(2, 8)$. Another method for solving simultaneous linear equations is Cramer's rule. Cramer's rule uses determinants to solve any system of n-variables and n-linear equations provided that the determinant of the coefficient matrix is not zero. Otherwise, the system is said to be singular; this means the system has infinite number of solutions. Using the previous example, we can use Cramer's rule to find the solution of the unknown variables provided that there are two unknown variables and two equations in this system and the determinant $|\boldsymbol{A}|$ is not equal to zero, as shown below:

$$\begin{cases} 14x - 2y = 12 \\ 3x + 2y = 22 \end{cases}$$

$$|\boldsymbol{A}| = \begin{vmatrix} 14 & -2 \\ 3 & 2 \end{vmatrix} = (14)(2) - (-2)(3) = 28 + 6 = 34.$$

The determinate is not zero, and the first step is to express the linear system in the following form $\boldsymbol{Ax} = \boldsymbol{b}$:

$$\begin{bmatrix} 14 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 22 \end{bmatrix}.$$

The procedure using Cramer's rule is to solve for one unknown variable at a time. The denominator is the determinant of the coefficient matrix. The numerator has the column matrix $\boldsymbol{b}$ replacing the column of the unknown variable. Solving for the unknown x-variable, replace vector $\boldsymbol{b}$ in the first column in the coefficient matrix and divide by the original coefficient matrix as follows:

$$x = \frac{\begin{vmatrix} 12 & -2 \\ 22 & 2 \end{vmatrix}}{\begin{vmatrix} 14 & -2 \\ 3 & 2 \end{vmatrix}} = \frac{(12)(2) - (-2)(22)}{(14)(2) - (-2)(3)} = \frac{24 + 44}{28 + 6} = \frac{68}{34} = 2.$$

The y–variable is found using this same process, but vector $\boldsymbol{b}$ replaces the second column instead of the first:

$$y = \frac{\begin{vmatrix} 14 & 12 \\ 3 & 22 \end{vmatrix}}{\begin{vmatrix} 14 & -2 \\ 3 & 2 \end{vmatrix}} = \frac{(14)(22) - (12)(3)}{(14)(2) - (-2)(3)} = \frac{308 - 36}{28 + 6} = \frac{272}{34} = 8.$$

The solution for the linear system is $(2, 8)$. Solving systems of simultaneous linear equations, three techniques have been demonstrated: substitution method, Gaussian elimination, and Cramer's rule. This ends the matrix section of the review, and a review of calculus rules starts the next section.

Calculus

While taking a calculus class, a student being able to take the derivative of a function is a simple process, but being able to recall all the differentiation rules after a five to ten-year lapse is perhaps painful! Students may spend countless hours relearning the differentiation rules while taking other classes that list calculus as a prerequisite. An area that may pose a problem for students is the different notations used when taking derivatives. There is more than one way to represent taking the derivative of a function (x) (read as f is a function of x or the shorter version is f of x). Table 1 has a list of popular differentiation notation that may be used.

Table 1

Differentiation Notation

Credited	function	1st	2nd	3rd	4th	nth
General	y	y'	y''	y'''	$y^{(4)}$	$y^{(n)}$
Lagrange's	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(n)}(x)$
Leibniz's	f	$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$	$\frac{d^3f}{dx^3}$	$\frac{d^4f}{dx^4}$	$\frac{d^nf}{dx^n}$
Newton's	y	$\dot{y}$	$\ddot{y}$	$\dddot{y}$	$\overset{4}{\dot{y}}$	$\overset{n}{\dot{y}}$

The notation (x) (i.e., $f(x) = 2x\ 2 + 3x - 8$) replaces the commonly used y (i.e., $y = 2x\ 2 + 3x - 8$) in function equation. The notation of y' is read as "y prime," the notation $f'(x)$ is read as "f prime is a function of x," and the notation $\frac{dy}{dx}$ is read as the "derivative of y with respect to x". When x and/or y is a function of time t, where time is the independent variable and x and y are the dependent variables, the dot notation is commonly used. Then the notation of a first derivative dot notation is $\dot{x}$ and $\dot{y}$; that means $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively. When starting to relearn the differentiation rules, start with the simplest rule to remember. Start with the easiest rule to recall, which is taking the derivative of a constant.

$\frac{d}{dx}[c] = 0$

$f(x) = 5$

$f'(x) = 0.$

The power rule is used to differentiate functions that can be expressed as a power of a variable. Generally, the power rule can be used to differentiate functions of the form (x) = u^n . Where u is a function of x. The rule is shown below:

$\frac{d}{dx}[u^n] = nu^{n-1}.$

An example of the power rule and the derivative of a function is shown below:

$(x) = x\ 5$

$f'(x) = 5x\ 5{-}1 = 5x\ 4$.

The constant multiplication rule is used when the constant may be pulled outside of the function as the rule shows:

$\frac{d}{dx}[cf(x)] = cf'(x)$

An example of pulling out the constant is shown below:

$f(x) = \frac{1}{2}x^3$

$f'(x) = \frac{d}{dx}\left[\frac{1}{2}x^3\right] = \frac{1}{2}\frac{d}{dx}[x^3] = \frac{1}{2}(3x^2) = \frac{3}{2}x^2.$

The sum or difference rule of two differentiable functions is the sum or difference of their derivatives. The rule for these two differentiable functions is as follows:

$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$ sum rule.

$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$ difference rule.

The product rule is used when differentiating the product of two functions. The product rule is as follows:

$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$

When using the product rule, it easier to placed brackets around the derivative and then take the derivative of the function, shown as follows:

$(x) = (x^2 + 2)(x^3 - 1)$

$f'(x) = (x^2 + 2)[x^3 - 1]' + (x^3 - 1)[x^2 + 2]'$

$f'(x) = (x^2 + 2)[3x^2] + (x^3 - 1)[2x]$

$f'(x) = 3x^4 + 6x^2 + 2x^3 - 2x.$

When there are no like terms to combine after taking the derivative of a function, the only thing to remember is writing the answer in descending order of exponents: $f'(x) = 3x^4 + 2x^3 + 6x^2 - 2x$. Using the quotient rule of derivatives, may be easier if brackets are placed around the derivatives. The quotient rule is listed below, followed by an example of this rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0.$$

The values $(h) = h^2$ and $f(h) = h^2 + 3$ plugged into the quotient rule as shown:

$$f(h) = \frac{h^2 + 3}{h^3}$$

$$f'(h) = \frac{(h^3)[h^2 + 3]' - (h^2 + 3)[h^3]'}{(h^3)^2}.$$

Simplify the equation as follows:

$$f'(h) = \frac{2h^4 - 3h^4 - 9h^2}{h^6}$$

$$f'(h) = \frac{-h^4 - 9h^2}{h^6}$$

$$f'(h) = \frac{-1}{h^2} - \frac{9}{h^4}.$$

The chain rule is used to differentiate the composite of two functions as given by the prime notation: $F = f \circ g$ (read as the function f of g) when both functions are differentiable. Then the derivative of the composite function is as follows: $F'(x) = f(g(x)) = f'(g(x))g'(x)$.

Take the derivative of the composite function f of g (listed below). Start by taking the derivative of the outside function; then take the derivative of the inside function as follows: $(x) = \sqrt{2 + x\,2.}$

Change the radical to an exponent; remember a fractional exponent is power divided by the root: $(x) = (2 + x^2)^{½}$.

Then apply the chain rule as follows:

$$F'(x) = \frac{1}{2}(2 + x^2)^{\frac{1}{2}-1}\,(2x)$$

$$F'(x) = \frac{1}{2}(2 + x^2)^{-\frac{1}{2}}(2x).$$

Multiply; then rewrite with positive exponents as follows:

$$F'(x) = \frac{2x}{2(2 + x^2)^{\frac{1}{2}}}.$$

Reduce and rewrite in radical form as shown:

$$F'(x) = \frac{x}{\sqrt{(2 + x)^2}}.$$

The chain rule may be combined with the general power rule into the definition listed as follows:

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x).$$

Rewrite the equation by moving the denominator into the numerator as follows:

$$y = \frac{3}{(x^4 + 2)^3}$$

$$y = 3(x^4 + 2)^{-3}.$$

Take the derivative of the function as follows:

$y' = 3\,(-3)(x^4+2)\,{-3-1} \cdot (4x^{4-1}\,)$.

The derivative of the outside function is $-3(x^4+2)^{-4}$, and the derivative of the inside function is $4x^3$. Multiply and rewrite using positive exponents as shown:

$$y' = \frac{-36x^3}{(x^4+2)^4}.$$

The derivatives of trigonometric functions are used when solving differential equations. Table 2 shows some of trigonometric functions needed when taking a differential equations class.

Table 2

Derivatives of Trigonometric Functions

Function	Derivative	Function	Derivative
$sin\,x$	$\cos\,x$	$\frac{d}{dx}[sin\,u]$	$(cos\,u)u'$
$cos\,x$	$-sin\,x$	$\frac{d}{dx}[cos\,\mathrm{u}]$	$-(sin\,u)u'$
$tan\,x$	$sec^2\,x$	$\frac{d}{dx}[tan\,u]$	$(sec^2\,u)u'$
$sec\,x$	$sec\,x\tan\,x$	$\frac{d}{dx}[sec\,u]$	$(sec\,u\,tan\,u)u'$
$csc\,x$	$-\csc x\cot x$	$\frac{d}{dx}[csc\,u]$	$-(csc\,u\,cot\,u)u'$
$cot\,x$	$-csc^2\,x$	$\frac{d}{dx}[cot\,u]$	$-(csc^2\,u)u'$

The trigonometric functions listed above are a few of the derivatives students will need to know when taking a differential equations class. Students will also need to recall how to integrate. The power rule for integration is as follows:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

A quick example of the power rule is shown below:

$\int 5x^3 dx = 5\left(\frac{x^{3+1}}{3+1}\right) = \frac{5}{4}x^4 + C$.

The general power rule for integration is another rule students need to study before starting a differential equation class. The function of the general power rule will need to be differentiable; the rule is listed below:

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C,\ n \neq -1.$$

The C is an integration constant of the function. This rule may also be applied to change a variable by choosing to substitution $\mathrm{u} = \mathrm{g(x)}$, which is as follows:

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1.$$

A simple example of the general power rule using both methods on the same problem is as follows:

$$\int (1+2x)^4 (2)dx.$$

The integrand can be written as $gx\ 4g'(x)$ where $g(x) = 1 + 2x$ and $g'(x) = 2$, and the general power rule of this function is as follows:

$$\int (1+2x)^4 2dx = \frac{(1+2x)^{4+1}}{4+1} + C = \frac{(1+2x)^5}{5} + C.$$

The process of change of variable by choosing to substitution $u = g\ (x)$ and $du = 2dx$ into the original integral as follows:

$$\int (1+2x)^4 (2)dx.$$

Make the substitution $u = 1 + 2x$ and $du = 2dx$ into the integral below:

$$\int u^4 du = \frac{u^{4+1}}{4+1} + C_1 = \frac{u^5}{5} + C_1.$$

Substitute $u = 1 + 2x$, but recall the integration constant must be different as follows:

$$\frac{(1+2x)^5}{5} + C_2.$$

Basic integration formulas that come up in a differential equations class that students need to make sure to study are located in Table 3.

Table 3

Basic Integration Formulas

Integral	Integral
$\int kf(u)du = k\int f(u)du$	$\int e^u du = e^u + C$
$\int du = u + C$	$\int \frac{du}{u} = ln\vert u\vert + C$
$\int[f(u) \pm g(u)]\, du = \int f(u)du \pm \int g(u)du$	$\int \sin u\, du = -\cos u + C$
$\int cos\, u\, du = sin\, u + C$	$\int tan\, u\, du = -ln\vert cos\, u\vert + C$
$\int f(ku)du = \frac{1}{k} F(ku) + C$	

These are just a few of the basic integral rules students should be familiar with before starting any course that calls for calculus as a prerequisite.

CHAPTER III

MODEL OF INTERACTING OF TWO POPULATIONS

Building a System

Building a constant population system of two competing species of x and y over a given area with respect to a given period of time t this assumes no interaction with other species can be challenging for students to comprehend. Students may have a better understanding of the meaning of each term of this system by first investigating how to build a logistic population model. The logistic population model assumes the growth rate of a certain single species in a given area with respect to a given period of time t, while a system investigates more than one species. When using this type of model, the important details that would be under consideration are the birth and death rates of the given species during this time and what the expected growth of the species would be when the population is small or large. The differential equation of a single population growth over a given period of time is shown as follows:

$$\frac{dp}{dt} = h(p)p.$$

The function $h(p)$ is the relative growth rate of the population, where h is the population birth rates minus the death rates for the unknown population p (where p is an arbitrary population). The population p size will be delegated by the growth rate of h. Using a small population size p, the growth rate h is expected to grow roughly at a constant speed. This is assuming that the species has all the necessities to thrive and that the birth rates (k) will outnumber the death rates (b). The function $h(p) = k - kp$ follows the property when b (a positive constant) is small; then h equates to the positive constant k, meaning $p < k\ b$.

When the population p grows too large for the resources available for the species to survive, the death rates will increase, and the birth rates will decrease. The logistic equation indicates when the population p is large, meaning $p > k,$ and then h will be negative as follows: $\frac{dp}{dt} = kp - bp^2$. The symbols k and b represent any arbitrary positive constants. The constants k and b will be replaced to simplify the process of building a competing system of two populations. The constant k will be represented by a_1 and a_2. This represents the birth rates of the populations x and y, while the constant b is represented by the constants b_1 and b_2 that denotes the death rates of these populations. The first species of the system will be population x, which will satisfy the function of the population growth with respect to time t over a given period as follows: $\frac{dx}{dt} = a_1x - b_1x^2$. The second species will be the population y, which will satisfy the function of the population growth with respect to time t over a given period as shown: $\frac{dx}{dt} = a_2y - b_2y^2$.

When neither of the species of x and y has any contact with the other, the system of the population growth of $x(t)$ and $y(t)$ will be the differential equation as follows:

$$\begin{cases} \frac{dx}{dt} = a_1x - b_1x^2 \\ \frac{dy}{dt} = a_2y - b_2y^2 \end{cases}.$$

A chance meeting of these two populations is proportional to the product xy. The completion system as seen below:

$$\begin{cases} \frac{dx}{dt} = a_1x - b_1x^2 - c_1xy \\ \frac{dy}{dt} = a_2y - b_2y^2 - c_2xy. \end{cases} \tag{1}$$

The coefficients of the last terms c_1 and c_2 will reflect the rate of the population decline, due to the frequency of these two population encounters. The coefficients a_1, a_2, b_1, b_2, c_1, and c_2 symbolize positive real numbers.

Investigation of Population Model

Start by investigating the population system of Equation 1 for any points of interest, critical values. We then investigated the movement surrounding these points for possible stabilization of each point. When the points are static (no movement) in its position, then this point is an equilibrium point of the system. Then investigate each equilibrium point for classification, the type of stability, and the state the possible outcome of the species of the populations in four case scenarios.

Equation 1 consists of two nonlinear equations and makes this system difficult to find an analytical solution. This is why we do the following:

$$\begin{cases} a_1x - b_1x^2 - c_1xy = 0 \\ a_2y - b_2y^2 - c_2xy = 0. \end{cases} \tag{2}$$

Start by locating the critical vales of the system by setting $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. next use Jacobian matrix to linearize the system near each critical point. Then find the local behavior of the model surrounding each critical point for a possible equilibrium point. The critical point will be found from the following system of equations. Factor the GCF of both equations in Equation 2; then set these equations to zero. The first equation shows that the GCF is the variable x, and in the second equation, the GCF is the variable y. The factored Equation 3 is as follows:

$$\begin{cases} a_1x - b_1x^2 - c_1xy = x(a_1 - b_1x - c_1y) = 0 \\ a_2y - b_2y^2 - c_2xy = y(a_2 - b_2y - c_2x) = 0 \end{cases} \quad (3).$$

Then, Equation 4 will be the system of equations with the GCF removed.

$$\begin{cases} a_1 - b_1x - c_1y = 0 \\ a_2 - b_2y - c_2x = 0 \end{cases} \quad (4).$$

There are four case where these equations are equal to zero. The first case is where $x = 0$ and $y = 0$. The second case is where $y = 0$ and $a_1 - b_1x - c_1y = 0$. The third case is when $x = 0$ and $a_2 - b_2y - c_2x = 0$. The fourth case is where $a_1 - b_1x - c_1y = 0$ and $a_2 - b_2y - c_2x = 0$ meaning in case one cannot have x as a divisor.

Four Case Scenarios

Case one. The x-variable in the equations $(a_1 - b_1x - c_1y) = 0$ is simplified as follows:

$$\frac{x(a_1 - b_1x - c_1y)}{a_1 - b_1x - c_1y} = \frac{0}{a_1 - b_1x - c_1y}$$

$$x = 0; \quad a_1 - b_1x - c_1y \neq 0.$$

The y-coordinate for the first critical point is found using the previous process. The solution for the y-variable in the second equation is as follows:

$$y(a_2 - b_2y - c_2x) = 0$$

$$\frac{y(a_2 - b_2y - c_2x)}{a_2 - b_2y - c_2x} = \frac{0}{a_2 - b_2y - c_2x}$$

$$y = 0; \; a_2 - b_2y - c_2x \neq 0.$$

The first equilibrium point $(0,0)$ (e.g., Figure 12).

Case two. The second case is where $y = 0$ is substituted in the first equation in Equation 4; then solve for the x-variable as follows:

$$a_1 - b_1x - c_1y = 0$$

$$a_1 - b_1x - c_1(0) = 0$$

$$a_1 - b_1x = 0.$$

The first step is to solve for the x-variable by adding the additive inverse of a_1 to both sides of the equation: $a_1 - b_1x + (-a_1) = 0 - a_1$. The second step is to isolate the x-variable by dividing both sides of the equation by the coefficient $-b_1$:

$$\frac{-b_1x}{-b_1} = \frac{-a_1}{-b_1}.$$

The third step is simplifying the equation: $x = \frac{a_1}{b_1}$ This will give the second critical (equilibrium) point $\left(\frac{a_1}{b_1}, 0\right)$ (e.g., see Figure 14).

Case three. The third case is where $x = 0$ is substituted in the second equation of Equation 4; the next step is to solve for the y-variable as shown:

$a_2 - b_2y - c_2x = 0$

$a_2 - b_2y - c_2(0) = 0$

$a_2 - b_2y = 0.$

The y-variable is then solved by adding the additive inverse of a_2 to both sides and then isolating the y-variable by dividing both sides of the equation by coefficient $-b_2$, which finally simplifies the equation as follows:

$a_2 - b_2y - a_2 = 0 - a_2$

$-b_2y = -a_2$

$\frac{-b_2y}{-b_2} = \frac{-a_2}{-b_2}.$

$y = \frac{a_2}{b_2}.$

This gives the third critical (equilibrium) point $\left(0, \frac{a_1}{b_1}\right)$ (e.g., see Figure 16).

Case four. Finding the fourth critical point is a tedious process, and it is more difficult than finding the previous three points. Start by using Equation 4 and writing these equations in the simplified system as follows:

$$\begin{cases} b_1x + c_1y = a_1 \\ c_2x + b_2y = a_2 \end{cases}. \tag{5}$$

Next, write Equation 5 in the matrix equation form of $\boldsymbol{Ax} = \boldsymbol{b}$ as shown:

$$\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{6}$$

The matrix equations may be solved by using a number of methods including the substitution method previously demonstrated, but Cramer's rule is used to solve for the last critical (equilibrium) point. The two restrictions of Cramer's rule include the following:

first, there must be the same amount of n-variables as n-linear equations, and then the second, the determinant of the coefficient matrix is not zero.

The first restriction is met, as there are as many equations as unknown variables. The second restriction, the determinant of the coefficient matrix $\boldsymbol{A}$ is not zero as shown:

$$|\boldsymbol{A}| = \begin{vmatrix} b_1 & c_1 \\ c_2 & b_2 \end{vmatrix} = b_1 b_2 - c_1 c_2.$$

When $|\boldsymbol{A}| \neq 0$, then Equation 5 will have a solution, thus Equation 1 will have four critical points. The x-coordinate and y-coordinate of the fourth critical point is solved by using Equation 6. The x-coordinate is as follows:

$$x = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & b_2 \end{vmatrix}}{\begin{vmatrix} b_1 & c_1 \\ c_2 & b_2 \end{vmatrix}} = \frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2}.$$

Then the y-coordinate is the following:

$$y = \frac{\begin{vmatrix} b_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} b_1 & c_1 \\ c_2 & b_2 \end{vmatrix}} = \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}.$$

This results in the fourth critical (equilibrium) point $\left(\frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2}, \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}\right)$ (e.g., see Figure 18). This case scenario simulates a peaceful coexistent population system where the x- and y-species cohabitate in a given area competing for the same resources during a given period of time. The process of finding the critical points are as shown: $(0,0)$, $\left(\frac{a_1}{b_1}, 0\right)$, $\left(0, \frac{a_2}{b_2}\right)$, and $\left(\frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2}, \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}\right)$ is completed. A more detailed investigation of the behavior surrounding each of these points is contained in the next chapter.

Jacobian Matrix

The linearization of differential equations by the Jacobian matrix is a process used to study the general behavior around special points of interest, called equilibrium (critical) points, to determine the stability surrounding this point of interest. The Jacobian matrix uses the partial derivative $\left(\frac{\partial}{\partial x}\right)$ and $\left(\frac{\partial}{\partial y}\right)$ of the elements in the matrix by the linearization of the systems of the differential equations in Equation 1 to a more manageable form. Recall Equation 1 represents the two arbitrary populations of x and y with respect to the independent variable of time t. This system may be written in dot notation of the first derivative $\dot{x}$ and $\dot{y}$, instead of using $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Then, Equation 7 will be as follows:

$$\begin{cases} \dot{x} = a_1x - b_1x^2 - c_1xy = F(x,y) \\ \dot{y} = a_2x - b_2y^2 - c_2xy = G(x,y) \end{cases}. \tag{7}$$

The linearized Jacobian matrix (Equation 8) is as follows:

$$\boldsymbol{J}_{(x,y)} = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} a_1 - 2b_1x - c_1y & -c_1x \\ -c_2y & a_2 - 2b_2y - c_2x \end{bmatrix}. \tag{8}$$

Jacobian matrix (Equation 8) is used to simplify Equation 7, which helps in find the characteristic equation and the eigenvalues of the Jacobian matrix at each equilibrium point.

Case One Equilibrium Point (0,0)

Case One demonstrates the implementation of Jacobian matrix (Equation 8) with the first equilibrium point $(0,0)$, where the variables of x and y are replaced by zero, shown as follows:

$$\boldsymbol{J}_{(0,0)} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

This is a square matrix that is also a diagonal matrix. A few facts apply to this type of matrices. First, a square matrix will have a characteristic equation. Second, any triangular matrices will have their eigenvalues located on the diagonal. The Jacobian matrix $\boldsymbol{J}_{(0,0)}$ is a triangular matrix. The eigenvalues are $\lambda_1 = a_1 > 0$, and $\lambda_2 = a_2 > 0$. Both eigenvalues are real and positive; then the point $(0,0)$ is unstable nodal point (see Appendix; e.g., see Figure 12) where both arbitrary populations will perish over time.

Nontraditional math students may find it hard to remember these two facts; therefore, the process for evaluating Jacobian matrix for each equilibrium point, and then locating the eigenvalues and characteristic equations, are demonstrated. This process is started by locating the eigenvalues. First, the Jacobian matrix $\boldsymbol{J}_{(0,0)}$ is subtracted by the identity matrix that has been multiplied by the scalar lambda (λ) as follows:

$$\boldsymbol{J}_{(0,0)} - \lambda \boldsymbol{I} = = \begin{bmatrix} a_1 - \lambda & 0 \\ 0 & a_2 - \lambda \end{bmatrix}.$$

Second, take the determinant of this matrix, and then set each of the parentheses equal to zero, and solve for both lambdas as follows:

$$\left|\boldsymbol{J}_{(0,0)} - \lambda \boldsymbol{I}\right| = (a_1 - \lambda_1)(a_2 - \lambda_2) = 0.$$

The solution for the λ_1 is shown here:

$$a_1 - \lambda_1 = 0$$

$$-\lambda_1 = -a_1$$

$$\lambda_1 = a_1.$$

The solution for the λ_2 is shown here:

$$a_2 - \lambda_2 = 0$$

$$-\lambda_2 = -a_2$$

$$\lambda_2 = a_2.$$

Third, the characteristic equation is the product of two binomials:

$(a_1 - \lambda)(a_2 - \lambda) = \lambda^2 - (a_1 a_2)\lambda + a_1 a_2$

This concludes the first case scenario.

Case Two Equilibrium Point $\left(\frac{a_1}{b_1}, 0\right)$

The point $\left(\frac{a_1}{b_1}, 0\right)$ is plugged into the Jacobian matrix. The y-variable is zero, and the x-variable is $\frac{a_1}{b_1}$ that is shown as follows:

$$\boldsymbol{J}_{\left(\frac{a_1}{b_1},0\right)} = \begin{bmatrix} a_1 - 2b_1\left(\frac{a_1}{b_1}\right) & -c_1\left(\frac{a_1}{b_1}\right) \\ 0 & a_2 - c_2\left(\frac{a_1}{b_1}\right) \end{bmatrix} = \begin{bmatrix} -a_1 & -\frac{c_1 a_1}{b_1} \\ 0 & a_2 - \frac{c_2 a_1}{b_1} \end{bmatrix}.$$

The solution of the Jacobian matrix is an upper triangular matrix. This means the eigenvalues are located on the diagonal of this matrix. The $\boldsymbol{J}_{\left(\frac{a_1}{b_1},0\right)}$ eigenvalues are $\lambda_1 = -a_1 < 0$ (negative number), and $\lambda_1 = a_2 - \frac{c_2 a_1}{b_1}$ can be positive if $a_2 > \frac{c_2 a_1}{b_1}$ or negative if $a_2 < \frac{c_2 a_1}{b_1}$, then the point $\left(\frac{a_1}{b_1}, 0\right)$ is either a saddle point or is asymptotically stable (see Appendix). The second method for solving for characteristic equations starts with the characteristic equation using the following formula: $\lambda^2 - tr\boldsymbol{J}\lambda + det\boldsymbol{J} = 0$. Start with the Jacobian matrix $\boldsymbol{J}_{\left(\frac{a_1}{b_1},0\right)}$, and fill in the formula as shown:

$$\boldsymbol{J}_{\left(\frac{a_1}{b_1},0\right)} = \begin{bmatrix} -a_1 & \frac{-c_1 a_1}{b_1} \\ 0 & a_2 - \frac{c_2 a_1}{b_1} \end{bmatrix}.$$

The characteristic equation is the following:

$$\lambda^2 - \left[a_1 + \left(a_2 - \frac{c_2 a_1}{b_1}\right)\right]\lambda + \left[(-a_1)\left(a_2 - \frac{c_2 a_1}{b_1}\right)\right] = 0.$$

Then the second method to solve for the eigenvalues uses the following formula:

$$\lambda_{1,2} = \frac{tr\boldsymbol{J} \pm \sqrt{tr\boldsymbol{J}^2 - 4det\boldsymbol{J}}}{2}.$$

Then the eigenvalues are as follows:

$$\lambda_{1,2} = \frac{\left[-a_1 + \left(a_2 - \frac{c_2 a_1}{b_1}\right)\right] \pm \sqrt{\left[-a_1 + \left(a_2 - \frac{c_2 a_1}{b_1}\right)\right]^2 - 4\left[(-a_1)\left(a_2 - \frac{c_2 a_1}{b_1}\right)\right]}}{2}.$$

The eigenvalues in simplified form are shown with numerical coefficients in the next section of this chapter. This will give the second critical (equilibrium) point $\left(\frac{a_1}{b_1}, 0\right)$, which is an unstable saddle point (e.g., Figure14). This concludes the second case scenario.

Case Three Equilibrium Point $\left(0, \frac{a_2}{b_2}\right)$

The point $\left(0, \frac{a_2}{b_2}\right)$ is substituted into Jacobian matrix where Jacobian matrix $\boldsymbol{J}_{\left(0, \frac{a_2}{b_2}\right)}$ is the x-variable that is substituted with zero and the y-variable is replaced by $\frac{a_2}{b_2}$. The reduced matrix is as follows:

$$\boldsymbol{J}_{\left(0, \frac{a_2}{b_2}\right)} = \begin{bmatrix} a_1 - c_1\left(\frac{a_2}{b_2}\right) & 0 \\ -c_2\left(\frac{a_2}{b_2}\right) & a_2 - 2b_2\left(\frac{a_2}{b_2}\right) \end{bmatrix} = \begin{bmatrix} a_1 - \frac{c_1 a_2}{b_2} & 0 \\ \frac{-c_2 a_2}{b_2} & -a_2 \end{bmatrix}.$$

This is a lower triangular matrix where the eigenvalues are located on the diagonal. The $\boldsymbol{J}_{\left(0, \frac{a_2}{b_2}\right)}$ eigenvalues are $\lambda_1 = a_1 - \frac{c_1 a_2}{b_2}$ that will be positive if $a_1 > \frac{c_1 a_2}{b_2}$ and will be negative if $a_1 < \frac{c_1 a_2}{b_2}$, then $\lambda_2 = -a_2 < 0$ (negative). This makes the point $\left(0, \frac{a_2}{b_2}\right)$ either a saddle point (determinant less than zero) or an asymptotically stable, where all the equilibrium solutions start near the equilibrium point and move toward the point as time increase.

The characteristic equation is shown below:

$$\lambda^2 - tr\boldsymbol{J}\lambda + det\boldsymbol{J} = 0$$

$$\lambda^2 - \left[a_1 - \frac{c_1 a_2}{b_2} - a_2\right]\lambda + \left[\left(a_1 - \frac{c_1 a_2}{b_2}\right)(-a_2)\right] = 0.$$

This concludes the third case scenario.

Case Four Equilibrium Point $\left(\frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2}, \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}\right)$

Case Four used a different method to locate the equilibrium point. The Cramer's rule was used to solve to find the x and y–coordinates of $x = \frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2}$, and $y = \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}$, where $b_1 b_2 - c_1 c_2 \neq 0$. A new method to solve for the eigenvalues and the characteristic equation is required. The reason for the new procedure is the determinant of the Jacobian matrix $\boldsymbol{J}_{\left(\frac{a_1b_2-c_1a_2}{b_1b_2-c_1c_2},\frac{b_1a_2-a_1c_2}{b_1b_2-c_1c_2}\right)}$ last term where Jacobian matrix is not triangular.

The first step in this process is substituting the equilibrium point into the Jacobian matrix, Equation 8 and simplifying. The x–variable in the Jacobian is replaced with the x–coordinate $\frac{a_1b_2-c_1a_2}{b_1b_2-c_1c_2}$, and the y–variable with the y–coordinate $\frac{b_1a_2-a_1c_2}{b_1b_2-c_1c_2}$. The simplified Jacobian matrix is as shown:

$$\boldsymbol{J}_{\left(\frac{a_1b_2-c_1a_2}{b_1b_2-c_1c_2},\frac{b_1a_2-a_1c_2}{b_1b_2-c_1c_2}\right)} =$$

$$\begin{bmatrix} a_1 - 2b_1\left(\frac{a_1b_2-c_1a_2}{b_1b_2-c_1c_2}\right) - c_1\left(\frac{b_1a_2-a_1c_2}{b_1b_2-c_1c_2}\right) & -c_1\left(\frac{a_1b_2-c_1a_2}{b_1b_2-c_1c_2}\right) \\ -c_2\left(\frac{b_1a_2-a_1c_2}{b_1b_2-c_1c_2}\right) & a_2 - 2b_2\left(\frac{b_1a_2-a_1c_2}{b_1b_2-c_1c_2}\right) - c_2\left(\frac{a_1b_2-c_1a_2}{b_1b_2-c_1c_2}\right) \end{bmatrix}.$$

An attempt to find for the trace and determinant of the matrix without the aid of computer programs is perhaps a herculean task for a nontraditional math student. Using the substitution method for the variables for each of the elements for the Jacobian matrix, Equation 8 will help simplify the process of finding the characteristic equation and the eigenvalues. Substitute the variables a, b, c and d into matrix $\boldsymbol{A}$ to ease the process as shown below:

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The trace and determinant of the matrix are simple to solve. The trace is: $tr\boldsymbol{A} = a + d$ and the determinant is: $|\boldsymbol{A}| = ad - bc$. The characteristic equation is the formula $\lambda^2 - tr\boldsymbol{A}\lambda + det\boldsymbol{A} = 0$ is as follows: $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$.

The process to solve for the eigenvalue is as follows:

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$= \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

This concludes the fourth case scenario.

Model of Population System

Equation 1 is used to investigate particular solutions of the four equilibrium points, the Jacobian matrix, and the characteristic equation by replacing the coefficients of a_1, a_2, b_1, b_2, c_1, and c_2 with the following numerical coefficients: $a_1 = 14$, $a_2 = 16$, b_1 and $b_2 = 2$, c_1, and $c_2 = 1$, thereby creating the following system:

$$\begin{cases} \frac{dx}{dt} = 14x - 2x^2 - xy \\ \frac{dy}{dt} = 16y - 2y^2 - xy \end{cases}. \tag{9}$$

Equation 9 represents two competing arbitrary population species of x and y over a given period of time t (Edwards, Penney, & Calvis, 2015). Locating the four equilibrium points for Equation 9 is substituting the numerical coefficients into the solutions of Equation 1. These equilibrium points of the four cases are as follows: $(0,0)$, $(7,0)$ where $\frac{a_1}{b_1} = \frac{\mathbf{14}}{\mathbf{2}} = 7$ and $(0,8)$ where $\frac{a_2}{b_2} = \frac{\mathbf{16}}{\mathbf{2}} = 8$.

The solution of the fourth equilibrium point was formed using the Equation 5 replaced by the numerical coefficients as follows:

$$\begin{cases} 2x + y = 14 \\ x + 2y = 16 \end{cases}. \tag{10}$$

The fourth equilibrium point was found by replacing the numerical coefficients into the solutions of x and y–coordinates of $x = \frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2}$ and $y = \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}$where $b_1 b_2 - c_1 c_2 \neq 0$.

This results in the following equilibrium point (4,6) with the x- and y-coordinates:

$$x = \frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2} = \frac{(14 * 2) - (1 * 16)}{(2 * 2) - (1 * 1)} = \frac{28 - 16}{4 - 1} = \frac{12}{3} = 4.$$

Then the y-coordinate is as follows:

$$y = \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2} = \frac{(2 * 16) - (14 * 1)}{(2 * 2) - (1 * 1)} = \frac{32 - 14}{4 - 1} = \frac{18}{3} = 6.$$

Replacing the coefficients in the Equation 7 as follows:

$$\begin{cases} \dot{x} = 14x - \ 2x^2 - xy = F(x, y) \\ \dot{y} = 16y - \ 2y^2 - \ xy = G(x, y) \end{cases}.$$

Jacobian matrix (Equation 8) with the numerical coefficients of Equation 9 is as follows:

$$\boldsymbol{J}_{(x,y)} = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 14 - \ 4x - \ y & -x \\ -y & 16 - 4y - \ x \end{bmatrix}. \tag{11}$$

The general solutions are used to solve the eigenvalues and characteristic equation of each of the four case scenarios using Equation 11.

Case One Equilibrium Point (0,0)

The first case equilibrium point $(0,0)$ is substituted for the variables x and y in the Jacobian matrix (Equation 11) as shown:

$$J_{(0,0)} = \begin{bmatrix} 14 - \ 4(0) - (0) & -(0) \\ -(0) & 16 - 4(0) - \ (0) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix}.$$

The eigenvalues of triangular matrices are located on the diagonal of the matrix and are $\lambda_1 = 14$ and $\lambda_2 = 16$.

The characteristic equation is written by finding the trace, $tr\boldsymbol{J}_{(0,0)} = 14 + 16 = 30$, and the determinant of matrix is as follows: $det\boldsymbol{J}_{(0,0)} = 14(16) - 0 = 224$ characteristic equation is: $\lambda^2 - 30\lambda + 224 = 0$. The corresponding eigenvectors for the equilibrium point (0, 0), where both populations will perish over a given period of time, was located using Maple (Version 10). These Maple data (in blue) list the eigenvectors in the order of $\lambda = 16$ first and then $\lambda = 14$ second, as shown in Figure 1.

$$[16, 1, \{[0 \quad 1]\}], [14, 1, \{[1 \quad 0]\}]$$

Figure 1. Maple solution eigenvector (0,0).

The equilibrium point $(0,0)$ is a proper nodal source since there is a repeated positive real eigenvalues with two linearly independent eigenvectors and the origin is unstable due to both the trace and determinant is greater than zero.

Case Two Equilibrium Point $(7,0)$

Case Two is where the equilibrium point $(7, 0)$ is substituted into the Jacobian matrix (Equation 11) as follows:

$$\boldsymbol{J}_{(7,0)} = \begin{bmatrix} 14 - 4(7) - (0) & -(7) \\ -(0) & 16 - 4(0) - (7) \end{bmatrix} = \begin{bmatrix} -14 & -7 \\ 0 & 9 \end{bmatrix}.$$

This is an upper triangular matrix, and the eigenvalues that are located on the diagonal are $\lambda_1 = -14$ and $\lambda_2 = 9$.

The characteristic equation is located by finding the $tr\boldsymbol{J}_{(7,0)} = -14 + 9 = -5$ and the determinant, $det\boldsymbol{J}_{(7,0)} = (-14)(9) - 0 = -126$. Then the characteristic equation is $\lambda^2 + 5\lambda - 126 = 0$. The eigenvectors for the second equilibrium point $(7, 0)$, are shown using Maple (Version 10) is shown in Figure 2.

$$[-14, 1, \{[1 \quad 0]\}], \left[9, 1, \left\{\left[1 \quad \frac{-23}{7}\right]\right\}\right]$$

Figure 2. Maple solution eigenvector (7,0).

The equilibrium point $(7, 0)$ is an unstable saddle point because the trace is less than zero, and the eigenvalues are real and unequal and have opposite signs.

Case Three Equilibrium Point (0,8)

Case Three shows the third equilibrium point $(0{,}8)$ is substituted into Jacobian matrix (Equation 11) as follows:

$$\boldsymbol{J}_{(0,8)} = \begin{bmatrix} 14 - \ 4(0) - 8 & 0 \\ -8 & 16 - 4(8) \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -8 & -16 \end{bmatrix}.$$

This is a lower triangular matrix, and the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -16$. The characteristic equation is located by finding the trace, $tr\boldsymbol{J}_{(0,8)} = 6 + (-16) = -10$ and the determinant, $\boldsymbol{J}_{(0,8)} = (6)(-16) - 0 = -96$. Then the characteristic equation is $\lambda^2 + 10\lambda - 96 = 0$. The Maple solutions for the eigenvectors of the equilibrium point $(0, 8)$ are shown in Figure 3.

$$\left[6,\ 1,\ \left\{\left[\ \frac{-11}{4}\quad 1\ \right]\right\}\right],\ [-16,\ 1,\ \{[0\quad 1\ ]\}\ ]$$

Figure 3. Maple solution eigenvector (0,8).

The equilibrium point $(0{,}8)$ is an unstable saddle point because the trace is less than zero, and the eigenvalues are real and unequal and have opposite signs.

Case Four Equilibrium Point $(\mathbf{4{,}6})$

Case Four shows that the fourth equilibrium point $(4{,}6)$ is substituted in the Jacobian matrix (Equation 11) as follows:

$$\boldsymbol{J}_{(4,6)} = \begin{bmatrix} 14 - \ 4(4) - (6) & -(4) \\ -(6) & 16 - 4(6) - \ (4) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix}.$$

The characteristic equation is located by the trace, $\boldsymbol{J}_{(4,6)} = -8 + (-12) = -20,$ and the determinant, $tr\boldsymbol{J}_{(4,6)} = (-8)(-12) - (-4)(-6) = 72$. The resulting characteristic equation is $\lambda^2 + 20\lambda + 72 = 0$. The eigenvalues are solved by using the following formula:

$$\lambda_{1,2} = \frac{tr\boldsymbol{J} \pm \sqrt{tr\boldsymbol{J}^2 - 4det\boldsymbol{J}}}{2}.$$

The trace and determinant are substituted into the formula and then simplified as shown:

$$\lambda_{1,2} = \frac{-20 \pm \sqrt{20^2 - 4 * 72}}{2} = \frac{-20 \pm \sqrt{400 - 288}}{2} = \frac{-20 \pm \sqrt{112}}{2}$$

The first lambda is solved as follows:

$$\lambda_1 = \frac{-20 + \sqrt{112}}{2}$$

$$\lambda_1 = -15.2915026.$$

The second lambda is solved below:

$$\lambda_2 = \frac{-20 - \sqrt{112}}{2}$$

$$\lambda_2 = -4.7084974.$$

The Maple (Version 10) solutions for the eigenvectors of the equilibrium point $(4,6)$ are shown in Figure 4:

$$\left[-10 + 2\sqrt{7}, 1, \left\{\left[1 \quad \frac{1}{2} - \frac{1}{2}\sqrt{7}\right]\right\}\right], \left[-10 - 2\sqrt{7}, 1, \left\{\left[1 \quad \frac{1}{2} + \frac{1}{2}\sqrt{7}\right]\right\}\right]$$

Figure 4. Maple solution eigenvector $(4, 6)$.

The equilibrium point $(4,6)$ is an improper nodal sink with distinct negative real eigenvalues and is unstable at the origin because the trace is less than zero, and the determinant is greater than zero.

CHAPTER IV

PPLANE GRAPH

Vocabulary

Students being able to make detail graphs, such as phase-plane portraits that include vector fields, solution curves, the nullclines, and equilibrium points of population models have the advantage of visually comparing the regions surrounding each equilibrium point. The graphs of solution curves help students to determine the classification as spirals, node, saddle, or center points. The graphs of nullclines help determine if the points are stable or unstable and if these points are an attractor, a repeller, or semi-stable. A student being able to comprehend the meaning of the previous statements requires him or her to know the vocabulary used, plus vocabulary not mentioned. This section is started by defining some of the pertinent vocabulary used when graphing a phase-plane portrait.

An investigation of a population model is done by location points (solutions) that are not changing (standing still) and investigate the regions surrounding each point. These points are called equilibrium points. Then we graph the equilibrium points and the trajectories (flow patterns) in an xy-plane, it is called a phase portrait. The trajectories represent the direction of motion over time.

The x-nullcline is where $\frac{dx}{dt} = 0$, and is found by solving for $f(x, y) = 0$. The points on the x-nullcline are represented by vertical arrows on the nullcline lines because this is a region where the population x does not change in size over time. The initial population on the x-nullcline we expect that there is no change in population x while population y may increase or decrease in size. The y-nullcline is where $\frac{dy}{dt} = 0$, and is found by solving for $g(x, y) = 0$. Using these arrows (flow of direction) on the nullcline lines, we can determine the stability of each equilibrium point. The classification of equilibrium solutions (points) includes possible spirals, nodes, saddle, and center points. The stability that is the movement surrounding each point is examined in the next section.

How to Use the PPLANE Program

John C. Polking's MATLAB based phase plane program of PPLANE (2005.10; www.math.rice.edu/~dfield) is used with a system of ordinary differential equations (ODEs) to plot vector fields, plot solution curves, plot nullclines, find equilibrium points, display eigenvalue and eigenvector information, and state the stability of the orbits around the equilibrium points. This program will classify the equilibrium solutions as spirals, node, saddle, and center points, and the stability of these points as stable or unstable (see Appendix). When students first open the PPLANE program, four display windows will show on the screen.

The first display window is the PPLANE Copyright window (see Figure 5), which needs to be read before the *Ok* button is clicked at the bottom. Please do not click the *Ok* button before reading this information.

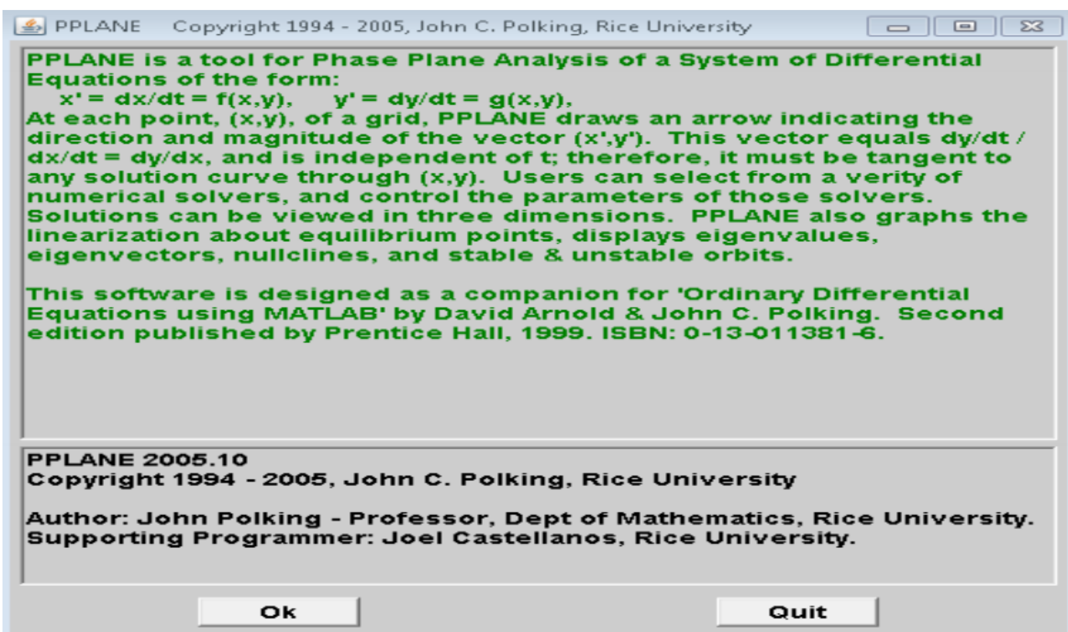

Figure 5. PPLANE copyright window.

This window will state what type of differential equations this program will solve and list some of the other features the program will compute.

The second window is the PPLANE Equation Window (see Figure 6). This is where students input their differential equations. The Equation 9 is used to demonstrate this program. Under the Gallery tab, there are seven other commonly used systems/equations that may be used to graph in this program. Some of the types of graphs include a linear system, a predator/prey, and the Van Der Pol's equation.

Another tab of interest is the Help tab that may be used to list the syntax and symbols for differential equations, which will open in the PLANE Message window. The Equation Window is set in the default form as follows:

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y).$$

This is the form used to graph the population system. Type the equations for the population system of x and y (see Figure 6). Spaces between the terms are allowed in the program language. Then type the dimensions of the graph in the display window for the graph size of the minimum and maximum of xy-plane. Enter the equation for the x-population after the x' box as $14x$ – $2x$^2 – xy; then enter the equation for the y-population as $16y$ – $2y$^2 – xy after the y' derivative box. The entry window will change red while the information is being entered.

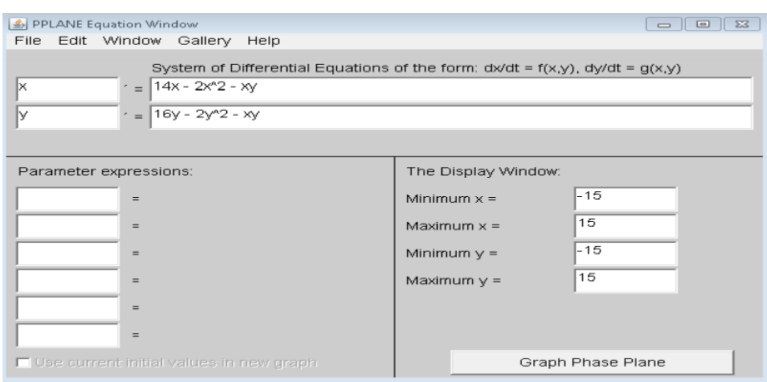

Figure 6. PPLANE equation window of population system.

Important note: If the equations are copied from Microsoft Word and pasted into the system window, an error message will show. This problem is fixed by retyping the minus signs. Next, set the Display Window of minimum values for the x/y variables -15 and the maximum values as 15.

Finally, click the Graph Phase Plane button to engage the third window, PPLANE Phase Plane (see Figure 7), the graphed trajectory of the system with green arrows.

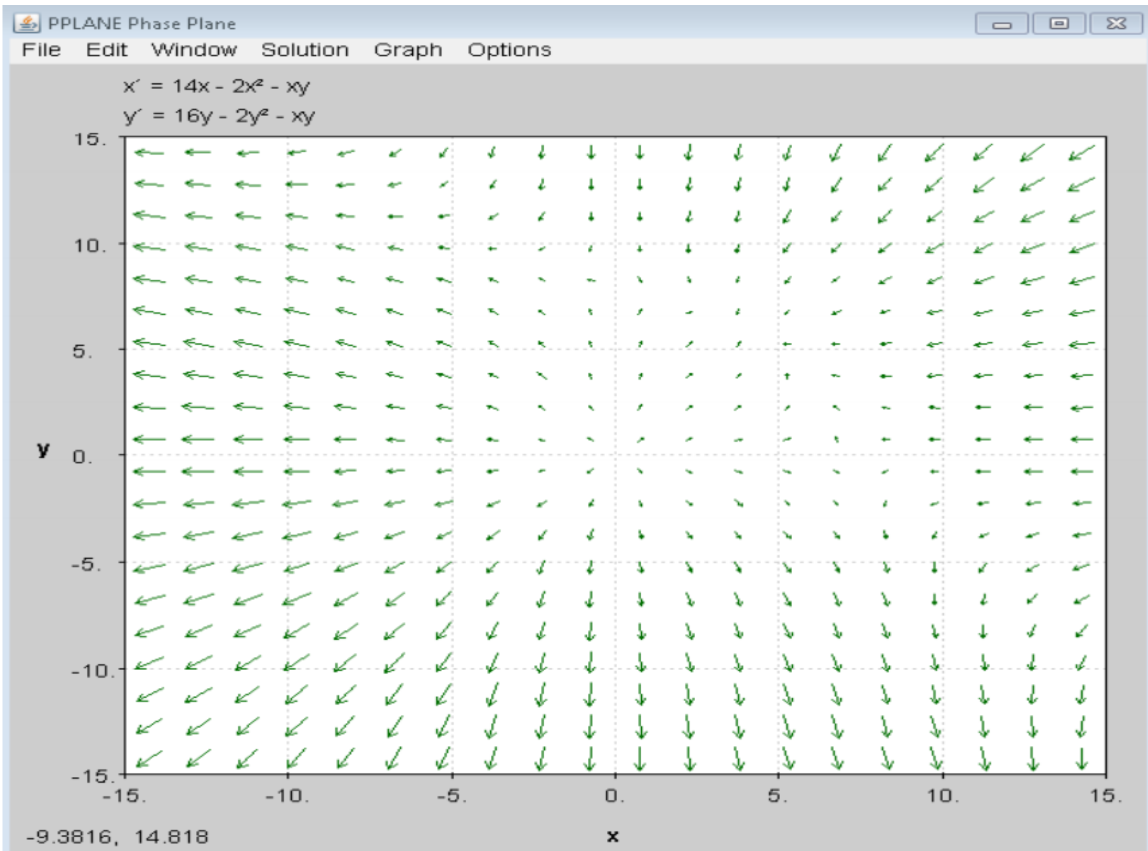

Figure 7. PPLANE phase plane graph of the system.

Finding the possible equilibrium points is easier if the screen is changed from the directional field to the nullcline plus arrow display of the system.

Change the screen, by clicking on the Solution tab, and then select Show Nullcline + Arrow option. The new screen, Figure 8, will show the nullclines of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vectors that are dividing the screen into increasing (positive) and decreasing (negative) slopes in the regions of the system. Where the nullclines intersect is an equilibrium point (red points). The x-nullcline (red lines) and the y- nullcline (yellow lines) detail the regions on the phase plane surrounding the equilibrium points to determine the stability of each point. Starting with the lower left-hand point $(0,0)$, the nullcline arrows move away or repel (called a repeller) from the equilibrium point. This point is an unstable point and is classified as a node source. The top left-hand point $(0,8)$ has the x-nullcline arrows moving toward and away from the equilibrium point and is considered semi-stable. The lower right-hand point $(7,0)$ has the y-nullcline towards x-nullcline arrows moving away and towards the equilibrium point and is also considered semi-stable. These semi-stable points are classified as saddle points. The top right-hand point $(4,6)$ has the x-nullcline and y-nullcline arrows moving towards the equilibrium point making this a stable attractor. This point is classified as a node sink:

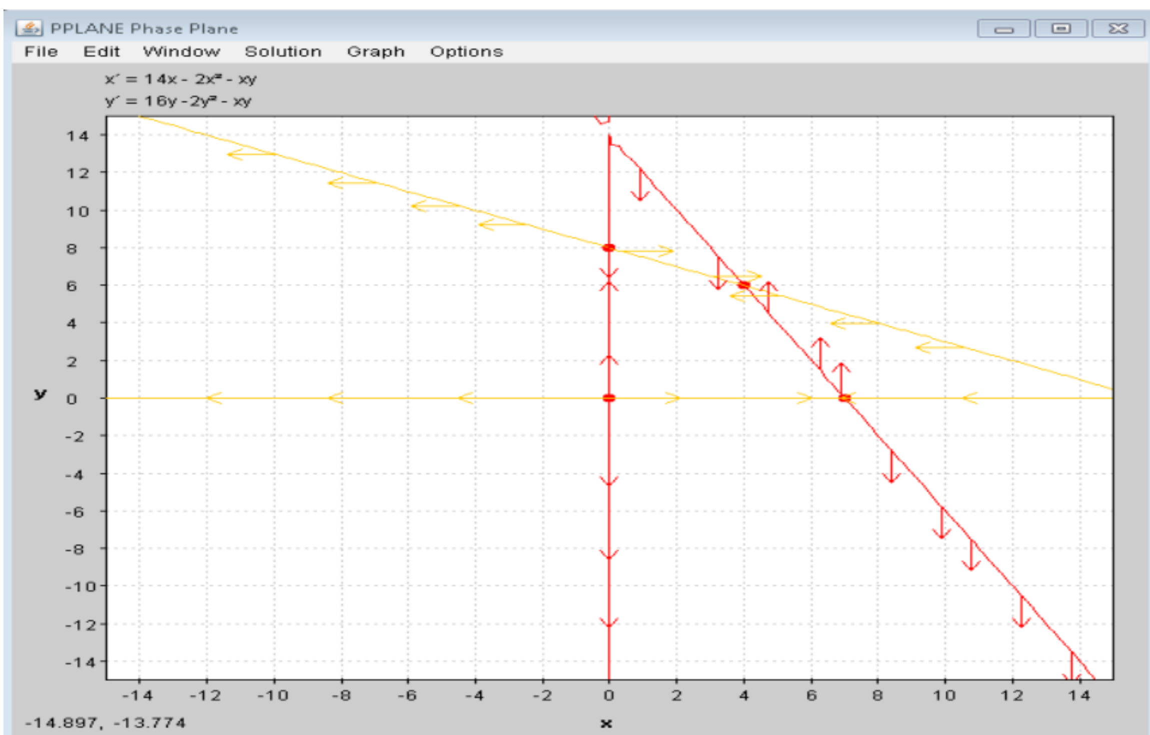

Figure 8. All four equilibrium points on nullcline and arrow graph.

Labeling the equilibrium points on the graph is accomplished by clicking the Edit tab and selecting Enter Text Annotation. Next, enter the point as $(0,0)$ that is going to be labeled. Then click *Ok*, and use the mouse to right-click on the graph where the point is to be labeled. Repeat this process for all four equilibrium points (see Figure 9).

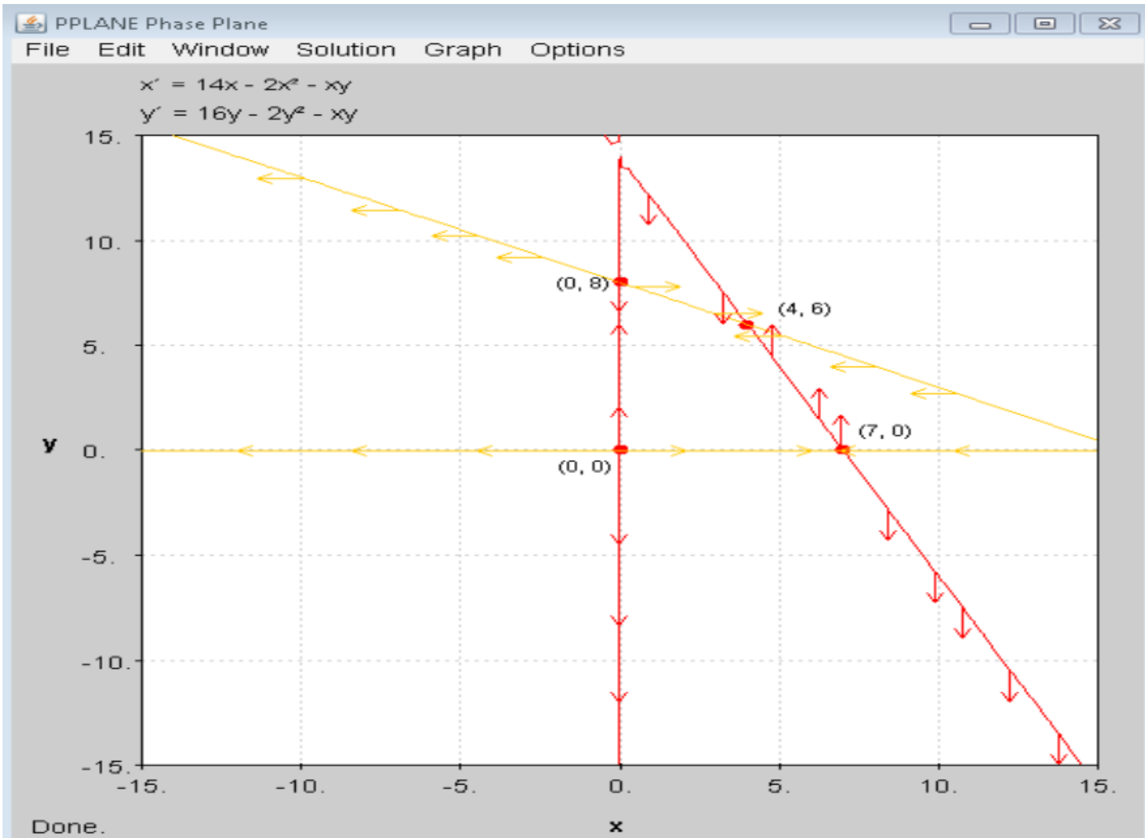

Figure 9. Labeled graph of four equilibrium points.

Placing the point labels incorrectly is easily fixed by clicking on the Edit tab and selecting either Erase Last Text Annotation (removes one) or Erase All Text Annotations to clear all the labels. The nullcline arrows on the phase portrait illustrate the slopes of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ the direction of growth or decay within a given bounded region. Using the bounded area in the center of the four equilibrium points, we will investigate the direction of the trajectories at four initial values from this area.

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The directions are located by evaluating a point in the $\frac{dx}{dt}$ and $\frac{dy}{dt}$ system. Evaluating the points $(1,1)$, $(1,7)$, $(3,5)$, and $(6,1)$ in Equation 9 yields the following components of the direction vectors in listed order:

$$\frac{dx}{dt} = 14(1) - 2(1)^2 - (1)(1) = 11$$
$$\frac{dy}{dt} = 16(1) - 2(1)^2 - (1)(1) = 13$$

$$\frac{dx}{dt} = 14(1) - 2(1)^2 - (1)(7) = 5$$
$$\frac{dy}{dt} = 16(7) - 2(7)^2 - (1)(7) = 7$$

$$\frac{dx}{dt} = 14(3) - 2(3)^2 - (3)(5) = 9$$
$$\frac{dy}{dt} = 16(5) - 2(5)^2 - (3)(5) = 15$$

$$\frac{dx}{dt} = 14(6) - 2(6)^2 - (6)(1) = 6$$
$$\frac{dy}{dt} = 16(1) - 2(1)^2 - (6)(1) = 8$$

Notice that the rate of change in x and y with respect to time is positive. This means that for points in this region the trajectory will have a positive slope. This population model means that any population of competing species, whose sizes are in this region, will be expected to move along the trajectory in the direction of growth. Finding a group of these direction vectors creates the trajectories on the phase portrait. Transiting to change the graph back to the direction field with nullclines (see Figure 10) is accomplished by clicking on the Solution tab. Then select the Show Nullclines, which will produce the same graph with the green arrows showing the direction vectors.

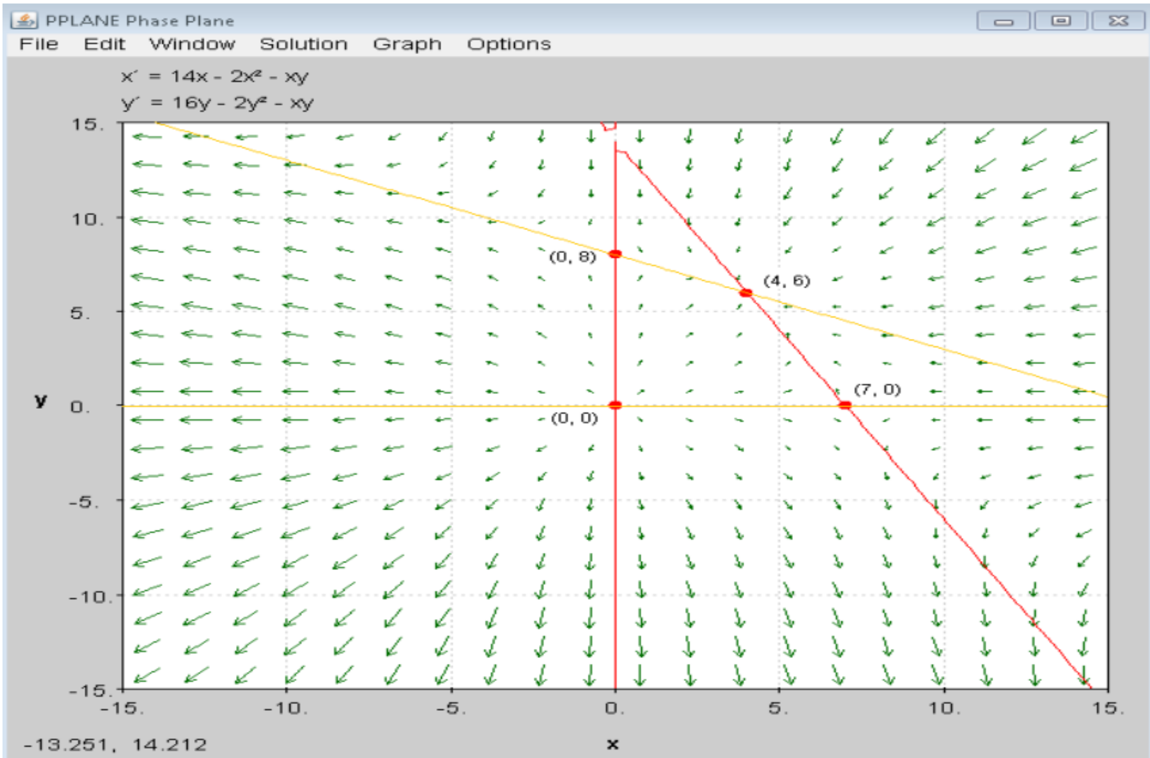

Figure 10. Direction field with nullcline and labeled points.

These direction vectors are tangent to the trajectories on a phase portrait and give the flow pattern in a given region of the solution curve. The bounded area between the equilibrium points shows the trajectories starting at the left-hand side moving up then over to the right. This will create the two nodal and the two saddle points. Creating a phase portrait including the solution curves, the equilibrium points, and the trajectories is the next step. The solution curves illustrates the behavior over time. We can graph the trajectories by clicking anywhere on the graph. These clicks represent a point on the trajectory.

The more clicks made, the easier it is to determine the classification of the equilibrium points–such as spirals, node, saddle, or center points. The lines of the phase portrait may obscure the four equilibrium points. This may be rectified by re-finding the equilibrium point over the previous found points (the red dots). The phase portrait (see Figure 11) shows all four of the equilibrium points and the trajectories of the entire graph. The zoom feature in PPLANE gives a clearer view of the stability surrounding each equilibrium point as an individual graph.

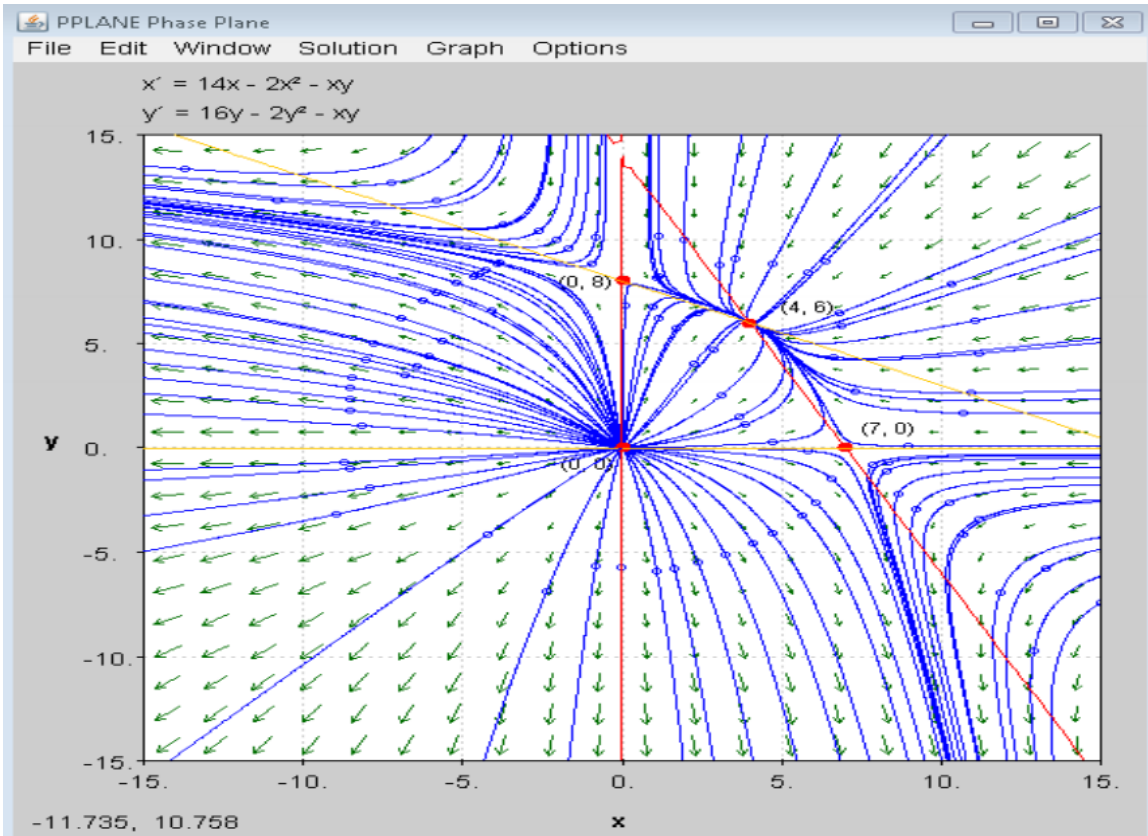

Figure 11. Phase portrait of the system.

Figure 11 details the solution curves as predicted. These curves start at the lower left in the region between the equilibrium points and move up and to the right. The two nodal and the two saddle points are easily seen. The solution curves may predict the population sizes of each species.

The equilibrium point $(0,0)$ is the trivial case in which the population and the rate of growth are null. The local region about the equilibrium point $(7,0)$ the system is unstable as seen in Figure 11. The local region about the equilibrium point $(0,8)$ the system is unstable as seen in Figure 11. By choosing any positive initial population in the region bounded by the four equilibrium point, say x_0 and y_0, then the point $(x(t), y(t))$ will approach the stable equilibrium point $(4,6)$ as $t \to +\infty$ as seen in Figure 11. This demonstrates the peaceful coexistence to both of populations. Both of the populations are small and the birthrates outnumber the death rates. The resources available are plentiful to sustain both populations.

Model Case One Graph

The zoom feature may be easier to use if the graph is completely cleared; then remake the phase portrait after the zoom graph is completed. However, it is not required. Clear the phase portrait by re-running the phase plane by clicking on the PPLANE Equation Window. Clicking the Graph Phase Plane button will clear the phase portrait, but not the point labels. Clear the point labels by selecting the Erase All Test Annotations under the Edit tab in the PPLANE Phase Plan window. Next, create a zoom graph of the first equilibrium point $(0,0)$.

The zoom graph is created by selecting the Edit tab and then by clicking on the Zoom-in: Select rectangle button. While the shift key is depressed, use the mouse to drag a rectangle box around the area to be defined. Release the mouse button, and the zoom graph will appear, as shown in Figure 12. The small blue circles (open dots on the graph) are clicks on the graph to make the phase portrait. The graph of Figure 12 shows that all the trajectory arrows (green arrows) are moving away from the equilibrium point $(0, 0)$, which is considered a nodal source since no spiraling has occurred.

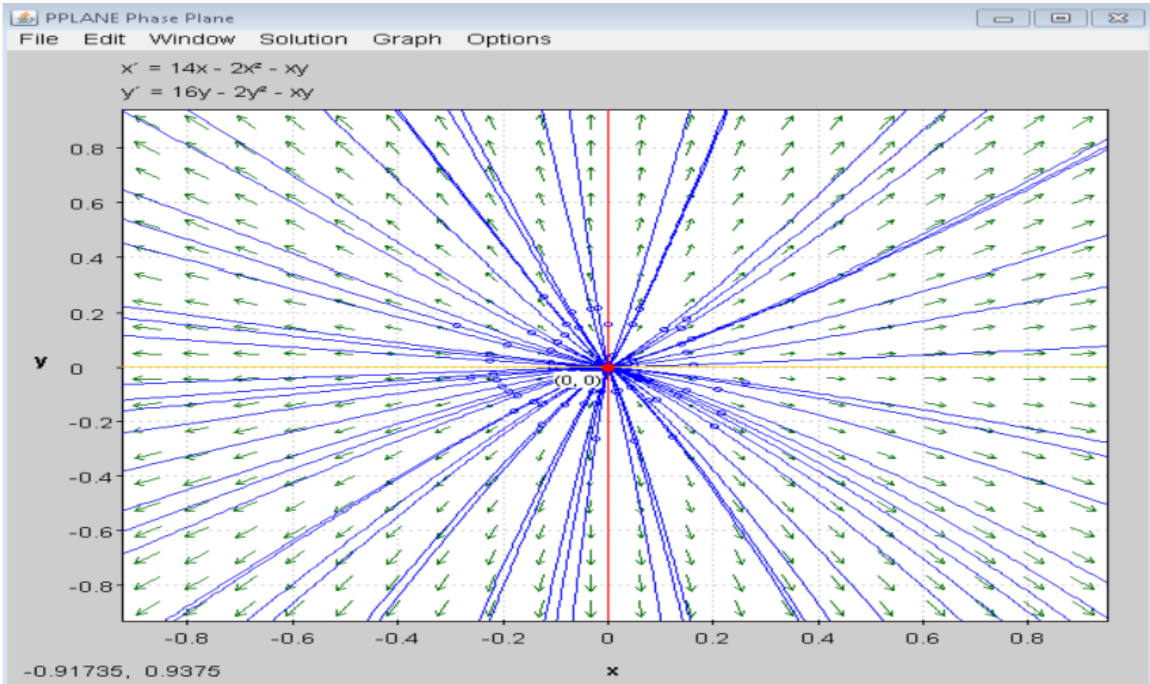

Figure 12. PPLANE graph point (0,0).

There are different types of nodal points. The nodal point is unstable (nodal source) when both eigenvalues are positive and is asymptotically stable (nodal sink) if both are negative. A proper nodal source has trajectories moving away from the point and has two repeated positive eigenvalues, which have two independent linear eigenvectors. An improper nodal source has either two distinct or two repeating positive real eigenvalues that will not have the two independent linear eigenvectors.

The equilibrium point $(0,0)$ has two eigenvalues that are real and distinct with like signs. This point is unstable and is an improper nodal source. The solutions for the eigenvalues and eigenvectors, along with the type of equilibrium point, may be seen in the PPLANE Messages window (see Figure 13).

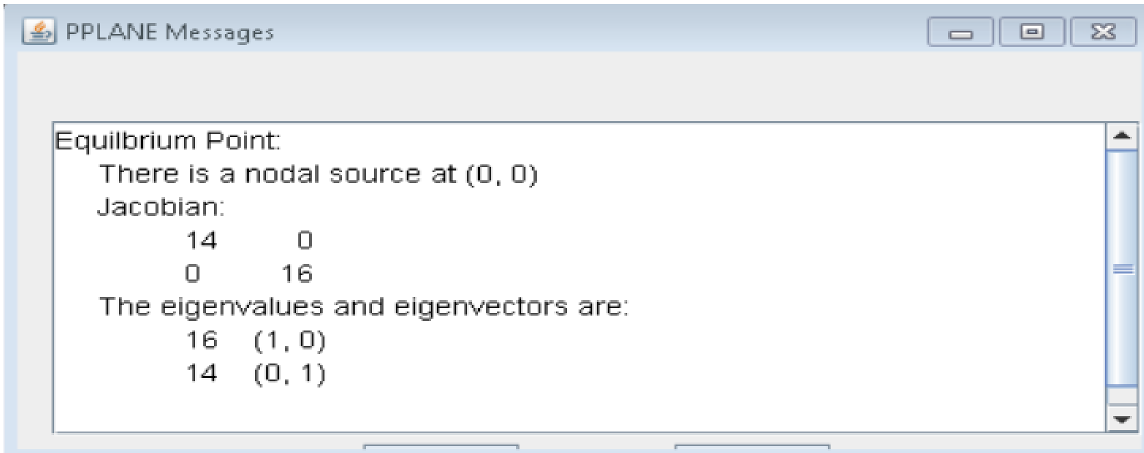

Figure 13. PPLANE messages window point (0,0).

Model Case Two Graph

The directions for making the zoom graph of Figure 8 should be followed to make the remaining equilibrium point zoom graphs. The equilibrium point (7,0) phase portrait (see Figure 14) shows the trajectory arrows (green arrows) moving both toward and away from the equilibrium point (7,0).

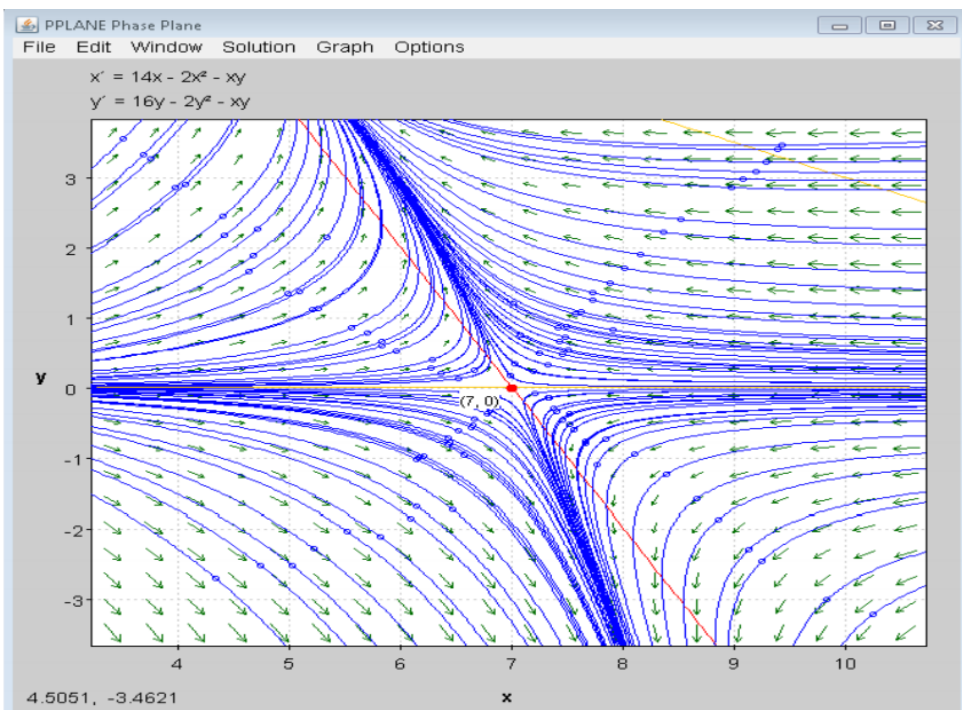

Figure 14. PPLANE graph point (7, 0).

Notice these trajectories will never cross the equilibrium point. The trajectory arrows starting from above the equilibrium point $(7,0)$ on the left side of the point, coming from negative infinity on x-axes moving to the left toward the equilibrium point then moving to positive infinity (on y-axis). The right side of the point, coming from positive infinity on x-axis moving left toward the equilibrium point then moving to positive infinity (on y-axis). The trajectory arrows from below the equilibrium point $(7,0)$ on the left side of the point, coming from negative infinity on x-axes moving toward the equilibrium point then moving to negative infinity (on y-axis). The right side of the point, coming from positive infinity on x-axis moving left toward the equilibrium point then moving to negative infinity (on y-axis). The point $(7,0)$ is an unstable saddle point because the eigenvalues are real and unequal and are opposite signs as seen in Figure 15.

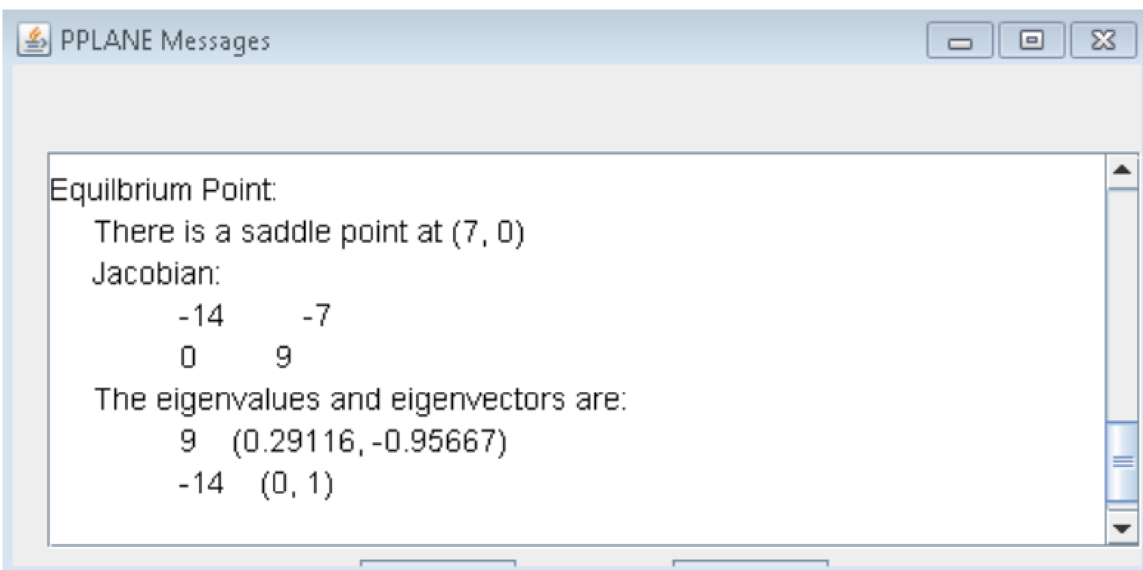

Figure 15. PPLANE Messages window point (7, 0).

Model Case Three Graph

The phase plane portrait of the equilibrium point $(0,8)$, Figure 16, shows the trajectories that again will never cross the point.

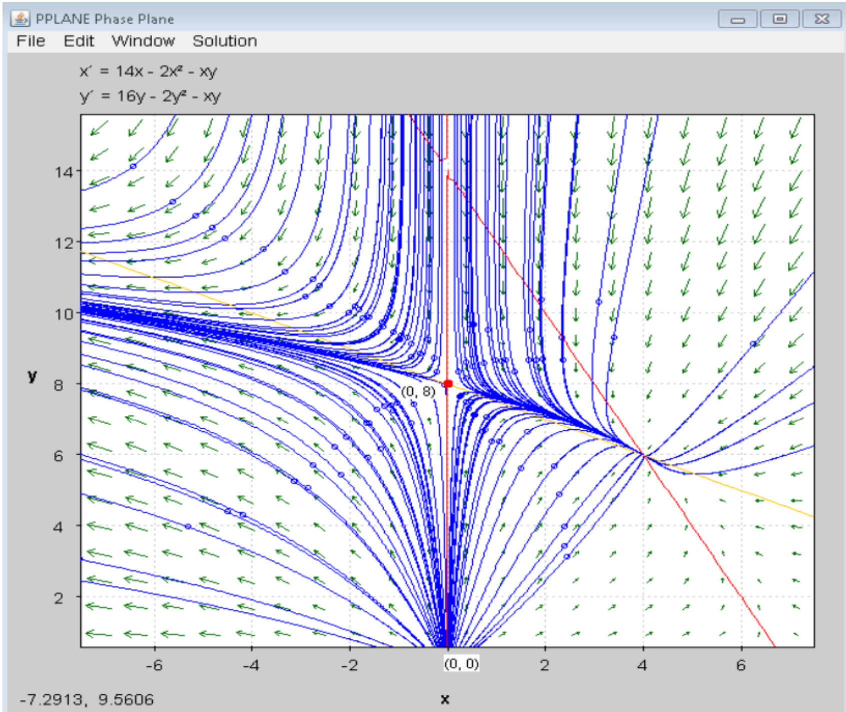

Figure 16. PPLANE graph point (0,8).

The trajectory arrows starting from above the equilibrium point (0,8) coming from negative infinity (on positive y-axes) and move toward the left to negative infinity and toward the right to positive infinity on the x-axis. The trajectory arrows from below the equilibrium point (0,8) start from negative infinity (on the y-axis) moving towards the equilibrium point then move toward the left towards negative infinity and toward the right to positive infinity on the x-axis. The equilibrium point (0,8) is an unstable saddle point with eigenvalues that are real and unequal and have opposite signs as seen in Figure 17.

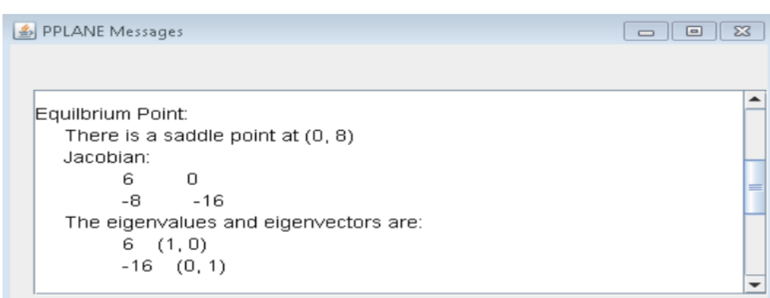

Figure 17. PPLANE Messages window point (0,8).

Model Case Four Graph

The zoom graph of the equilibrium point $(4,6)$, Figure 18 has the trajectories moving towards the point.

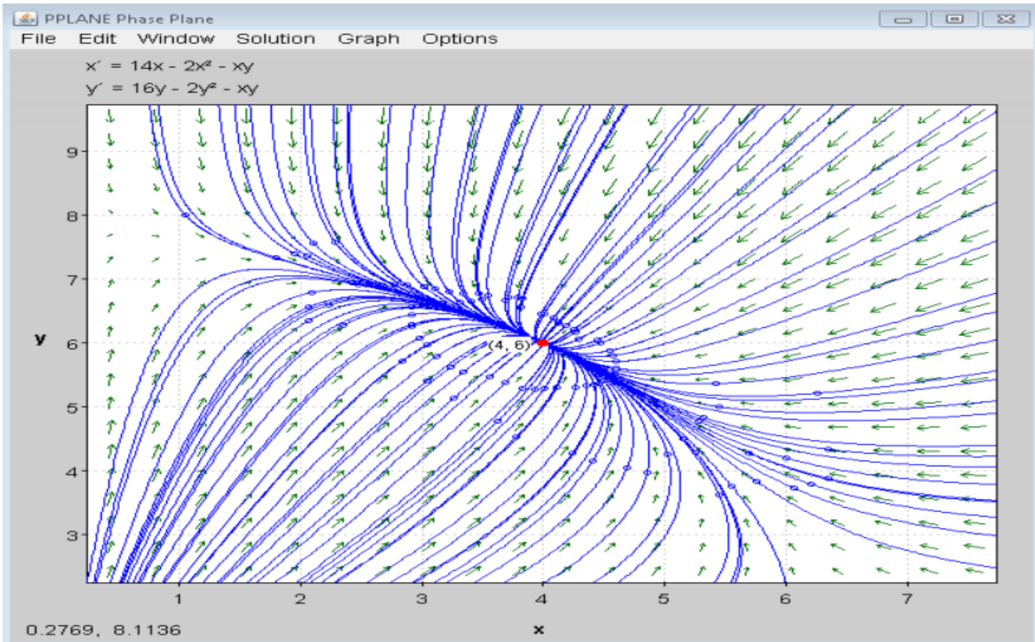

Figure 18. PPLANE graph point (4, 6).

Check the PPLANE Messages window (Figure 19) output data. Both eigenvalues are real, and both have negative signs with two independent linear eigenvectors. This makes the equilibrium point $(4,6)$ a stable nodal sink. Every trajectory (green arrows) is approaching the equilibrium point making this point asymptotic stable.

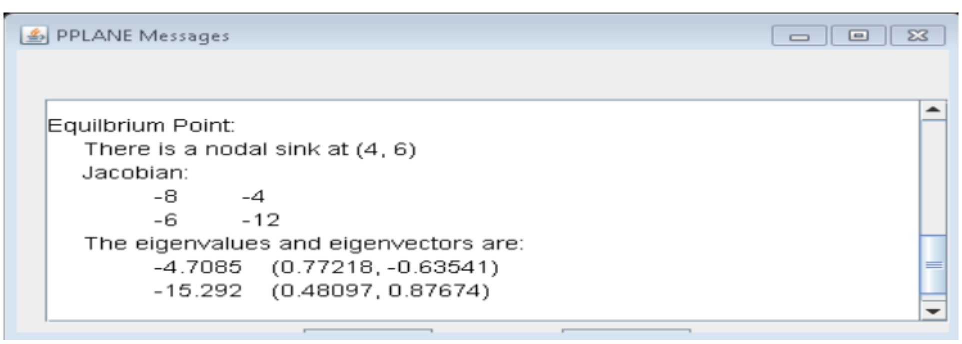

Figure 19. PPLANE Messages window point (4,6).

The equilibrium point $(4,6)$ is a proper nodal sink that has the trajectories moving towards the point and has two independent linear eigenvectors with two negative real eigenvalues. This concludes the investigation of the population model of two species.

CHAPTER V

CONCLUSION

An extended period of time is perhaps a nontraditional college math student's worst enemy. The mathematical concepts learned in one course become the prerequisites required in future courses. Postsecondary institutions offer students help understanding their assignments in the form of math labs. Nontraditional students have time restraints due to work and family schedules and may find it difficult to seek help during lab hours. Nontraditional students using online web sources to search mathematical definitions may find it troublesome, since the online search for ODE (ordinary differential equation) retrieved about 28,500,000 results in 0.92 seconds. The online computer searches, however, cannot distinguish between an ordinary differential equation and a lyric poem called an ode, meaning that finding online math help is only as good as the user.

Having a mathematical guide that contains examples of the prerequisite mathematical concepts used in future courses, including a section on how to use the math tools (computer programs) and at least one specific example from the new course is information that students have hands-on access to even late at night. More research is needed to investigate how much of the mathematical concepts students lose due to prolonged period in completing their degree, if a mathematical guide, such as this, will help improve students' chances of obtaining degrees.

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APPENDIX

Vocabulary

Vocabulary

Asymptotically stable - the all the equilibrium solutions that starts near the equilibrium point and moves toward the point as time increase.

Center point - an equilibrium point where the trace of a matrix is greater than zero and the determinant equals zero. The trajectories will orbit around the point.

Eigenvalues - the characteristic roots are the critical points (values) of polynomial characteristic equations.

Eigenvectors - a trajectory from the initial condition (usually with respect to the independent variable of time which equals zero) of equation in the form of the eigenvalues that will multiply to the eigenvector to lengthen the vector in the same direction of the initial condition of the vector.

Equilibrium point - a stationary point in a system which the x or y values have no change.

Isoclines – the equation of $y' = f(x, y)$ is the one-parameter family of curves within a plane, given by $f(x, y) = m$, where m is the slope and it is a constant.

Node - indicates no spiraling has occurred.

Nullclines – provide a skeleton picture of the changes at different points in a plane. The x-nullcline is the region of point where $f(x, y) = 0$. The y-nullcline is the region of points where $g(x, y) = 0$.

Phase-plane portrait - the possible solutions within a given dimensions of a Cartesian plane (an x/y plane) and the lines tangent to the phase trajectory over a given period of time for differential equations.

Saddle point- an equilibrium point has the determinant of a matrix is less than zero.

Sink or source - the discriminate of a matrix is less than zero, then the equilibrium point is a stable spiral or an unstable spiral, respectively.

Solution curves - the trajectory of a system in a Cartesian plan (an x/y plane).

Stable points - a sink where all the trajectories will move towards the equilibrium point.

Unstable points - a source where all the trajectories will move away from the equilibrium point.

Vector field graphs - groupings of arrows that represent short line segments (direction and magnitude) which are the approximate solutions of a system of differential equation.