

OPTIMAL CONTROL OF PREDATOR-PREY MODELS

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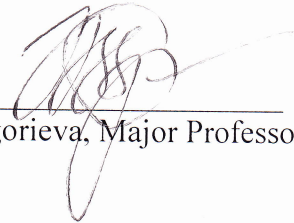
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To the Dean of the Graduate School:

I am submitting herewith a thesis written by Brandy Morrissey entitled "Optimal Control of Predator-Prey Models." I have examined this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science with a major in Mathematics.




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We have read this thesis and recommend its acceptance:







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Accepted:



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ABSTRACT

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A review of some models used to model ecological predator-prey populations is summarized. The Lotka-Volterra, Holling Logistic, and Holling-Tanner models are analyzed through linearization with the Jacobian to characterize the equilibrium points, and computer simulations are obtained to observe the trends of the models' trajectories. Additionally, an analysis of papers on optimal control of predator-prey models is presented to show a few different focuses that have already been utilized in the field of optimal control. Finally, an optimal control problem with two bounded controls for the Holling-Tanner predator-prey model is stated and solved. A computer simulation of that control model at selected values of the parameters and several objective functions is also presented. Furthermore, applications of such a control model are discussed in the context of real world predator-prey situations.

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CHAPTER I

INTRODUCTION

There are over 1000 animal species around the world that are endangered at varied levels - creating a deepening importance for monitoring the populations of natural wildlife. As the human population increases, natural habitats are being affected through destruction and environmental changes, which in turn affects the populations of wildlife, including predator-prey communities. To understand the progression of predator-prey native environments involves understanding many issues, such as predator-prey interactions and population dynamics among others.

The natural predator-prey interactions that exist between many species have been studied extensively. Many studies attempt to study locations that are as untouched by human activities as possible to observe these complicated relationships. Despite outside factors that influence predator and prey actions, some dominant behaviors typically exist between sets of interacting species. A primary behavior exhibited by prey species is migration to another area for a reduction in predation danger. For instance, some species will abandon an asset, such as a good feeding area, to attain a safer environment. However, predators usually respond to this alteration in

prey behavior with the intent of increasing their own chance of survival. Consequently, the predator-prey interaction can become like a behavioral game.

This behavioral game model has been studied by several mathematical ecologists. Behavioral ecologist Larouche studied one instance of the behavioral game model by examining the interactions between white sharks and Cape fur seals [1]. In this study, adult seals displayed behavioral strategies in response to deviations from sharks. In contrast, the sharks' behaviors were influenced by the location of seal pups. Additionally, marine biologist Bjorndal conducted a study of tiger sharks in Australia which showed that tiger sharks alter the habitat selection of green sea turtles because of threatening proximity [2]. The sea turtles relocate to an area of lesser quality sea grass for feeding to reduce the risk of predation. These examples reflect some of the behavioral game interactions that can occur between predator-prey populations.

When studying the population dynamics of the growth or decline of a single population, factors such as the natural growth rate and the carrying capacity, or limiting amount, of the environment are taken into consideration. Since mathematical ecology requires the study of the interaction of populations, each population affects the other's growth and mortality rates. Predator-prey population dynamics are often dramatic, such as the Canada lynx and snowshoe hare, in Canadian taiga (forest) biome. The Hudson's Bay Company provided the best long-term data set through records of

numbers of animals trapped, showing the fluctuations of lynx and hare populations across Canada, as shown in Figure 1.1 [3].

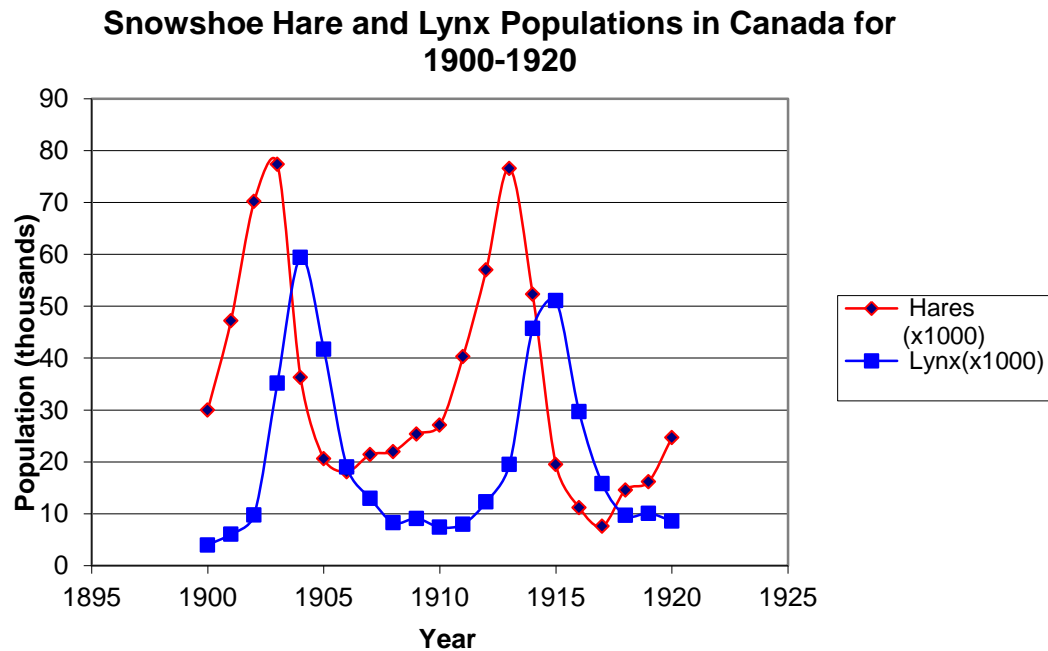


Figure 1.1: Snowshoe Hare and Lynx Populations in Canada for 1900-1920

An analysis of the dynamics of the two populations shows a correspondence in population values. The snowshoe hare's population fluctuates in cycles with an eight to ten year periodicity with a close following of the lynx cycle. Observing the data for the hares, there is a peak of the population, where perhaps there is more food, followed by a decline in the population, and then another peak about ten years after the first. The lynx data shows a similar periodicity, with an offset of one to two years. The minimum population values correspond to very low populations of hares, indicating that there was not enough prey to sustain the population of lynx. These data have caused many

mathematicians and scientists to question why the constant periodicity occurs. English zoologist and animal ecologist Charles Elton published a paper in 1924 which was first of many publications to analyze this data set [4]. Speculations such as hare-resource interaction, human interaction, reproduction rate decline of hare and lynx populations, and climate changes have been considered to explain the population cycles and changes.

In order to predict, describe, and explain such population changes in a species, mathematical models are developed, studied and improved. Such models are capable of predicting future patterns of many situations. Developments in science and technology have enabled people to increase the predictability of changes in natural populations and guide changes. Many mathematicians and scientists have modeled situations of various predator-prey groups with different differential equations. These models can be used to make predictions about the behavior of a number of different predator-prey populations, and provide clues as to why they behave as they do. As some populations near extinction, it becomes necessary for biologists and ecologists to look at manipulations of such populations to determine if populations can be recovered through human intervention. Therefore, the equations used for modeling become important to analyze and predict the populations to eliminate extinction.

Predation models are some of the oldest in ecology, including Bazykin's work on nonlinear predator-prey model dynamics [5], Volterra's data analysis of Adriatic fish [6],

Holling's studies with insects [7], and Tanner's investigations with various predator-prey situations [8]. During the course of this thesis, some predator-prey models will be reviewed and analyzed. Additionally, an analysis of papers on optimal control of predator-prey models will be presented to show the different focuses that have already been utilized. Finally, an optimal control problem with two bounded controls for the Holling-Tanner predator-prey model will be stated and solved. A computer simulation of that control model at selected values of the parameters and several objective functions will be presented. Furthermore, applications of such a control model will be discussed in the context of real world predator-prey situations.

CHAPTER II

REVIEW OF PREDATOR PREY MODELS

Applications of mathematics to ecology have been a source of curiosity to mathematicians for hundreds of years. Mathematicians have proposed models to describe the populations and still others have studied these models, honing them for particular situations. For instance, Bazykin studies many nonlinear models, such as the Lotka Volterra Model and Holling Model, extensively and outlines three main types of interactions between species that are recognized in ecology: mutualism or symbiosis, mutual competitive suppression or competition for a common resource, and predator-prey or parasite-host interactions [5]. He develops a set of general differential equations to model these interactions:

$$\begin{cases} \dot{u} = A(u) - B(u, v) \\ \dot{v} = -C(v) + D(u, v) \end{cases}$$

where $A(u)$ is the rate of increase of prey in absence of predators, $C(v)$ is the rate of mortality of predators in absence of prey, $B(u, v)$ is the rate of predation, and $D(u, v)$ is the rate of reproduction of the predators. Bazykin outlines types of functions to model interactions between predator and prey.

Bazykin concludes that there are some stabilizing factors that lead to a globally attracting equilibrium and other destabilizing factors that cause the model's trajectories to go to infinity. The stabilizing factors Bazykin notes are competition among prey, competition among predators for prey or other resources, and nonlinearity in the trophic function, $B(u, v)$, when the prey population density is small. Destabilizing factors include predator saturation and nonlinear reproduction of the predator and prey populations at small densities. Simultaneous consideration of two or more stabilizing factors or two or more destabilizing factors does not lead to new observations. Therefore, combinations of stabilizing and destabilizing factors would need to be studied to produce new equilibria or other results of stabilization or destabilization. Bazykin considers combinations of each factor and outlines them.

Other mathematicians, such as Berryman [9], have also outlined the various models associated with predator-prey population predictions throughout the years. Since such models are capable of providing feedback about population levels, many ecologists and mathematicians have studied the history of such models. Some of the various models that have been used will be described in the next section.

2.1 Malthusian Model. Thomas Malthus was the first to recognize a definitive theoretical treatment of population dynamics in 1798 [5]. Naturally, the first model typically introduced to estimate population growth is his model, known as the

Malthusian model, where the rate of change in the population is proportional the population,

$$P' = kP$$

where $P' = \frac{dP}{dt}$ and k is the proportionality constant. This is a separable differential equation and is easily solved to get $P = Ae^{kt}$. To find a particular solution representing either the predator or prey population, it would be necessary to utilize data to find the particular values for constants A and k . However, this model continues to grow exponentially, as observed in Figure 2.1, and does not realistically describe populations in the wild, since most populations are limited by resources.

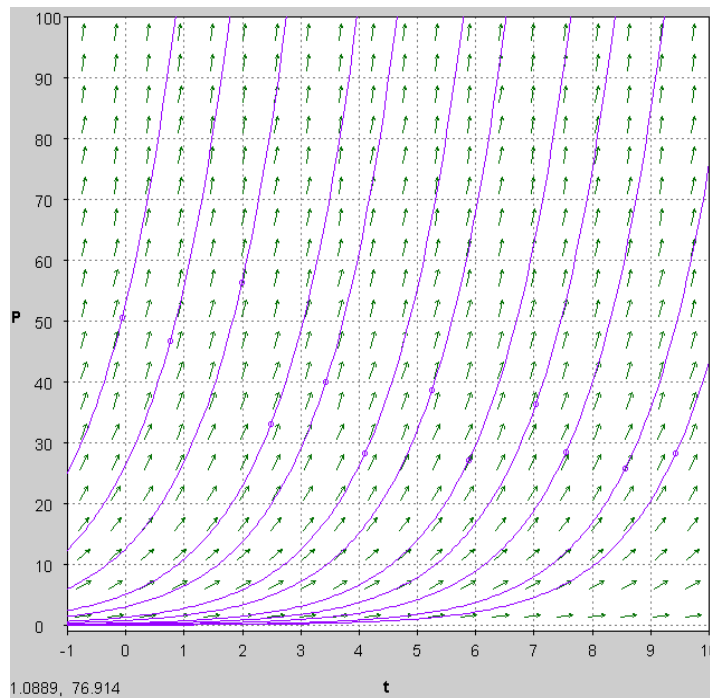


Figure 2.1: Malthusian Model of Exponential Growth, $k = 0.75$

2.2 Verhulst-Pearl Model. Verhulst described the dynamics of a population that was restricted in size by some limited resources first by the equation

$$\frac{dx}{dt} = \frac{ax(k-x)}{k}$$

now known as the logistic equation, in 1838 [5]. a is the rate of exponential population growth at small population size, while k is the carrying capacity of the population density, determined by the available resources. Pearl rediscovered this work in 1927 after it was forgotten. This model grows exponentially as the Malthusian model at first, but reaches a point where the rate of growth is a maximum and then the rate of growth begin to decline as shown in Figure 2.2.

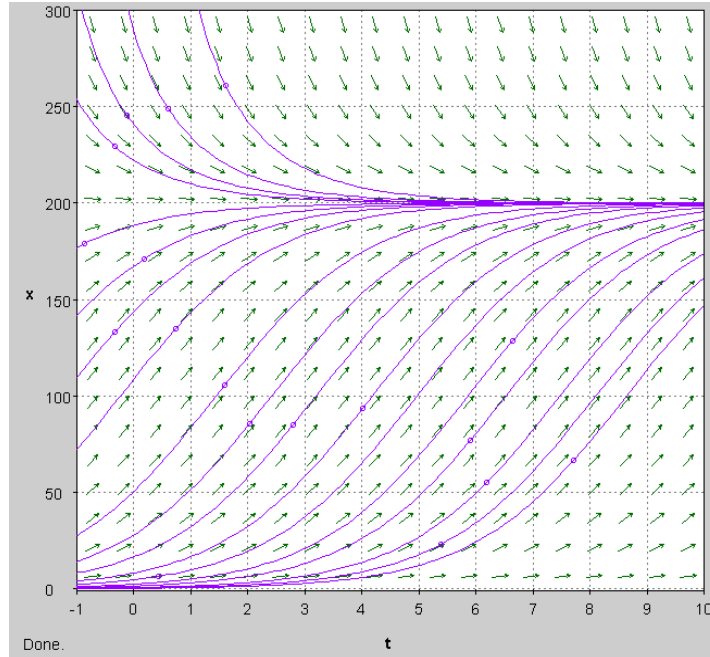


Figure 2.2: Verhulst-Pearl Logistic Model, $a = 0.75$, $k = 200$

If a population were to become larger than the carrying capacity, for instance with the introduction of new offspring or animals via human transport, then the population would decline and approach the carrying capacity because of the limited resources. However, this model still does not address the many interactions between predator and prey, such as competition for resources and satiation of the predator. From a mathematical point of view when describing two species, one equation cannot model all of the interactions and dynamics exactly. Therefore, multiple equations are advantageous since such equations could model the populations of the predator-prey better.

2.3 Lotka-Volterra Model. Umberto D’Ancona, an Italian biologist, studied the variations in population density of interacting fish species in the Mediterranean following World War I. His data give the average percentage of total catch of selachians (sharks, skates, etc) for 1914-1923, shown in Table 2.1 [6].

Table 2.1: Average Percentage of Total Catch for Selachians for 1914-1923

Year	1914	1915	1916	1917	1918	1919	1920	1921	1922	1923
Percent Catch of Selachians	11.9%	21.4%	22.1%	21.2%	36.4%	27.3%	16.0%	15.9%	14.8%	10.7%

The data show a large increase in the percent selachian catches for 1915-1922.

D’Ancona reasoned that the reduced fishing activity during the war accounted for this

abnormality. This reasoning led him to question the affects of the intensity of fishing on fish populations [6]. This anomaly was also important to the fishing industry, since it would influence fishing practices.

D'Ancona theorized that reduced fishing for food-fish gave rise to an increased food-fish population, and therefore, an increased selachian population. However, this theory does not explain why reduced fishing is more beneficial to sharks than food-fish. D'Ancona consulted with Vito Volterra on this situation [6].

Vito Volterra investigated a wide range of ecological problems, including the effects of migration and of several species simultaneously interacting and developed a predator-prey model to help explain the fish studies of D'Ancona. Independently and concurrently, Alfred Lotka initially proposed a predator-prey model, extended the model to organic systems using a plant and herbivore as an example, and utilized his equations to analyze predator-prey interactions. The subsequent model describing two species predator-prey and competition models are called the Lotka-Volterra model in honor of both the biologist and mathematician who developed the basic model. The pair of equations is a set of first order, nonlinear differential equations:

$$\begin{cases} \frac{dx}{dt} = ax - bxy = x(a - by) \\ \frac{dy}{dt} = -cy + dxy = -y(c - dx) \end{cases}$$

where x is the number of prey, a is the prey population growth rate, b is the attack rate of predators, y is the number of predators, c is the rate of predator decline in absence of prey, and d is the predator population growth rate due to predation.

The Lotka-Volterra Model makes several assumptions about the environment and predator-prey evolutions, including that the prey population finds ample food at all times, that the food supply of the predator population depends entirely on the prey populations, that the rate of change of population is proportional to its size, and that during the process, the environment does not change in favor of one species, and genetic adaption is slow.

2.4 The Fishing Activity Modified Lotka-Volterra Model. D'Ancona concluded that wide variations in fishing activities during WWI caused the variation in the percent increase of selachian catch [6]. To address the fishing effects, Volterra modified his equations with several thoughts in mind. First he reasoned that fishing activity decreased food-fish populations at a rate ex where $e > 0$ and reflects the intensity of fishing activity. Secondly, he concluded that fishing activity decreases selachian populations at a rate of ey . So he modified his equations:

$$\begin{cases} \frac{dx}{dt} = ax - bxy - ex = (a - e)x - bxy \\ \frac{dy}{dt} = -cy + dxy - ey = -(c + e)y + dxy \end{cases}$$

As long as $a - e > 0$, the model has a structure similar to the previous model.

2.5 Holling Logistic Model. C.S. Holling modified the Lotka-Volterra system with the

logistic equation $\frac{dx}{dt} = r\left(1 - \frac{x}{k}\right)x$, where k is the carrying capacity of the population

density, with the conclusion that it models a single prey population better [7]. Let x'

denote $\frac{dx}{dt}$. When x is small, there is very little interspecies competition for resources,

$1 - \frac{x}{k}$ is close to 1, and the reproductive rate per individual $\frac{x'}{x} = k\left(1 - \frac{x}{k}\right)$ is very close to

k . As x increases, interspecies competition increases, $1 - \frac{x}{k}$ approaches 0, and the

reproductive rate per individual approaches 0. Using this logistic model as the model

for the prey population and keeping the predator population model, the model

becomes

$$\begin{cases} \frac{dx}{dt} = a\left(1 - \frac{x}{k}\right)x - bxy \\ \frac{dy}{dt} = -y(c - dx) \end{cases}$$

2.6 Holling-Tanner Model. The predation rate increases with increasing prey density as

studies by Holling and others have shown, until the predation rate reaches a limiting

capacity. Holling introduced the function $y = w\frac{x}{r + x}$ to satisfy the predation rate when

predators will not or cannot kill more prey even when preys are available [7]. James

Tanner did extensive studies with this refinement to determine its application [8]. w

and r are constants that determine how fast the functional response increases at low densities of the prey. Increasing or decreasing the value of r increases or decreases the rate at which the predation function approaches its limit. r is interpreted as the predator search time. w is the maximum predation rate (i.e. the predation limit as x approaches infinity).

With this new model for predation, the revised system becomes

$$\begin{cases} \frac{dx}{dt} = a\left(1 - \frac{x}{k}\right)x - yw\frac{x}{r+x} \\ \frac{dy}{dt} = -y(c - dx) \end{cases}$$

This still leaves the predator model as the original. Since the revised prey model more accurately reflects how the prey population growth rate responds to the particular predator and prey combinations, the predator equation should also be revised to reflect the predator growth and decline rates. $\frac{dy}{dt} = -cx + dxy$ implies that as the prey population approaches infinity, the predator growth rate approaches infinity as well. Natural growth rate assuming infinite resources would better model this situation. If the predator population is less than the natural carrying capacity defined by limiting the resources, the predator population would grow at a rate between zero and the maximum. If the predator population is more than the natural carrying capacity defined by limiting resources, the predator population would decline. These observations suggest that the predator population is similar to the logistic equation introduced for

prey. However, the limiting resource in predator population is considered a function of the prey population. These considerations introduce the model for predators as

$$\frac{dy}{dt} = c \left(1 - \frac{y}{j} \right) y = c \left(1 - \frac{y}{j} \right) y, \text{ where } j \text{ is the number of prey required to support one}$$

predator at equilibrium.

Now the model becomes

$$\begin{aligned} \frac{dx}{dt} &= a \left(1 - \frac{x}{k} \right) x - yw \frac{x}{r+x} \\ \frac{dy}{dt} &= c \left(1 - \frac{y}{j} \right) y \end{aligned}$$

Other mathematicians, such as May, also did work independently regarding this model and its applications [10].

CHAPTER III

ANALYSIS OF MODELS

In addition to creating the model mathematically, the model should be analyzed to determine not only if it simulates the situation well, but also to determine overall characteristics of the solutions curves to gain an understanding of the changes in populations. Different methods such as the use of nullclines, the process of linearization using the Jacobian, or numerical methods using computer programs such as Maple and/or Matlab can be used to observe features and trends.

3.1 Analysis: Nullclines and Linearization. The nullclines, or zero-growth isoclines, of a

system of differential equations are the geometric shape for which $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

The equilibrium points of a system are located where the nullclines intersect. The

trends of the solution curves can be observed by looking at the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in

the sections that the nullclines bound.

Population equilibrium occurs when neither population level is changing, so

$\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. When the system of differential equations is solved, equilibrium

points are obtained. The stability of each point can be determined by analyzing the

Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Each equilibrium point can be analyzed individually using the characteristic equation to determine the eigenvalues to produce the stability conclusion.

3.2 Lotka-Volterra Model Analysis. The Lotka-Volterra model

$$\begin{cases} \frac{dx}{dt} = ax - bxy = x(a - by) \\ \frac{dy}{dt} = -cy + dxy = -y(c - dx) \end{cases}$$

has been analyzed by many mathematicians, such as Braun [6], to find equilibrium points and trends of trajectories.

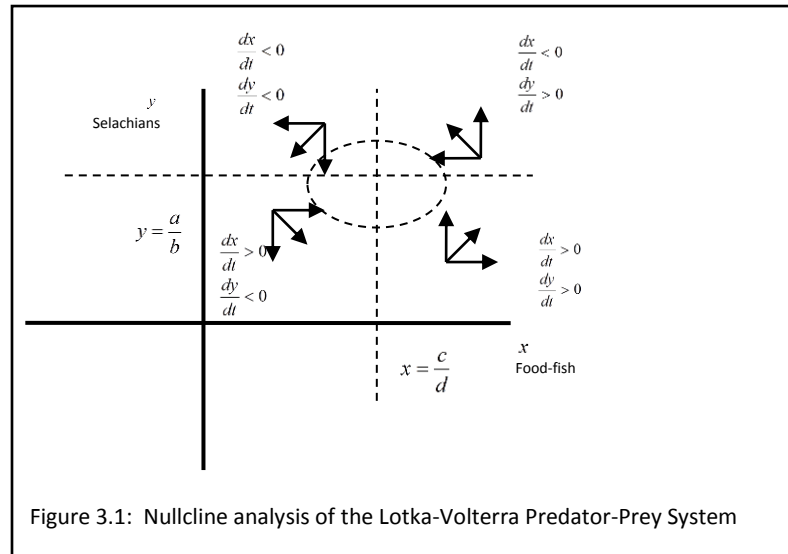
For prey $\frac{dx}{dt} = 0$ so $0 = x(a - by)$. Since a and b are constants the zero growth

nullcline for prey is a constant, defined by predator numbers, $y = \frac{a}{b}$. For predators

$\frac{dy}{dt} = 0$ so $0 = -y(c - dx)$. Since c and d are constants, the zero nullcline for predators is

also a constant defined by prey numbers, $x = \frac{c}{d}$. An analysis of the signs of $\frac{dx}{dt}$ and

$\frac{dy}{dt}$ leads to the conclusions in Figure 3.1 below.



Population equilibrium occurs when neither population level is changing, so

$\frac{dx}{dt} = 0 = x(a - by)$ and $\frac{dy}{dt} = 0 = -y(c - dx)$. When solved two equilibrium points are

obtained: $(0,0)$ and $\left(\frac{c}{d}, \frac{a}{b}\right)$.

The stability of each point can be determined by analyzing the Jacobian matrix:

$$J(x, y) = \begin{bmatrix} a - by & -bx \\ dy & dx - c \end{bmatrix}$$

For the fixed point at the origin,

$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

producing the characteristic equation $\lambda^2 - (a - c)\lambda - ac = 0$, with eigenvalues a and $-c$. Since $a > 0$ and $c > 0$, the eigenvalues have opposite signs, implying that $(0,0)$ is a saddle point. This unstable equilibrium point represents the point of extinction of both species. If it were stable, non-zero populations could be attracted towards it and lead to extinction.

For the second fixed point $\left(\frac{c}{d}, \frac{a}{b}\right)$,

$$J\left(\frac{c}{d}, \frac{a}{b}\right) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$$

producing the characteristic equation $\lambda^2 + ac = 0$, with eigenvalues $\pm i\sqrt{ac}$. Since the eigenvalues are purely imaginary, closed trajectories surround the point on the phase portrait. The equilibrium solution is a center and is neutrally stable or just stable.

Therefore, levels of predator-prey oscillate around the point.

The Lotka Volterra models predator-prey situations more realistically. Analyzing the dynamics of the model, observations show that the predators thrive when there is plentiful prey, but ultimately, outstrip their food supply and decline. As the predator population is low, the prey population will increase again. These dynamics continue in a cycle of growth and decline, as shown in Figures 3.2 – 3.4. However, in the absence of predators, the prey population still grows according to the Malthusian model.

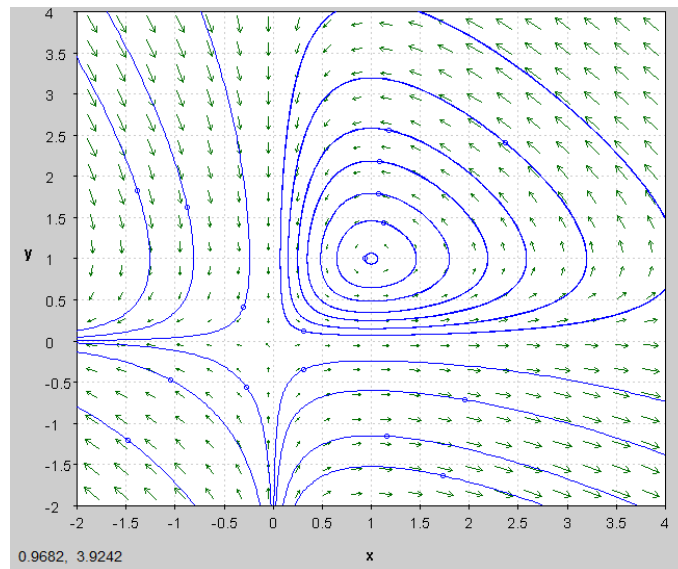
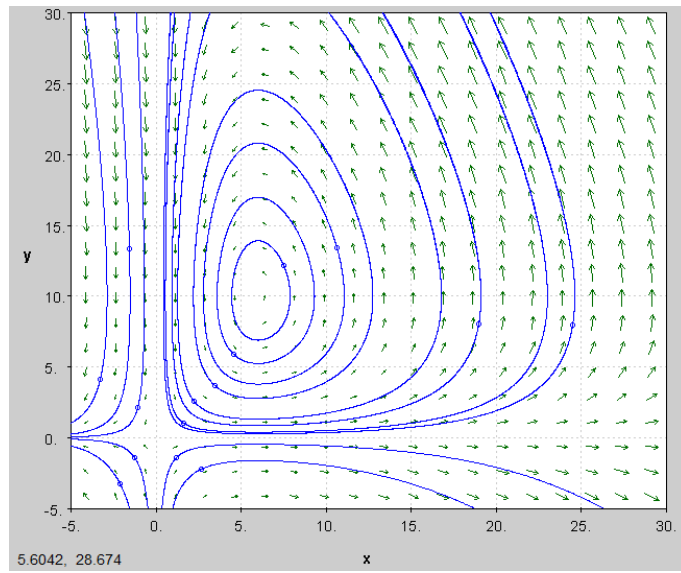
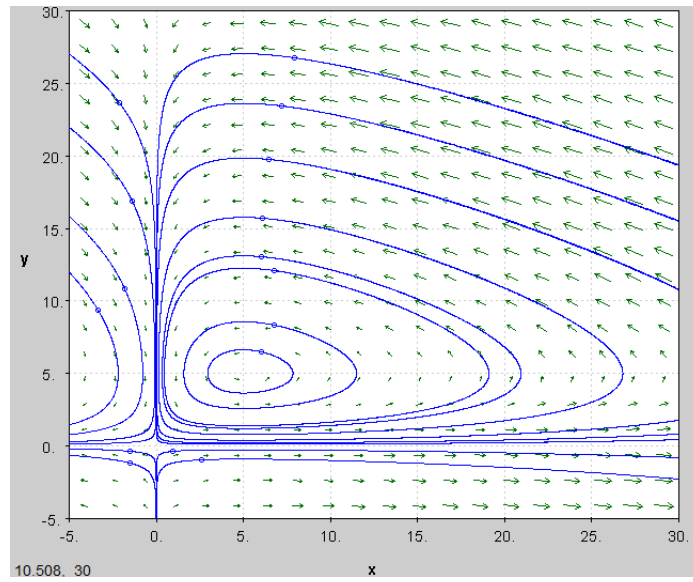


Figure 3.2: Phase portrait with $a = b = c = d = 0.3$



Analyzing the dynamics of the Lotka-Volterra model, observations show that although predators thrive in conditions with abundant prey, ultimately, their population will decline once the food supply dwindles. Once the predator population reaches its minimum, the prey population will increase again. These dynamics continue in a cycle of growth and decline, shown in figure 3.5 using the same values of parameters as shown in figure 3.2, and show a similar pattern as the lynx-hare empirical data collected by the Hudson's Bay Company as seen in figure 1.1.

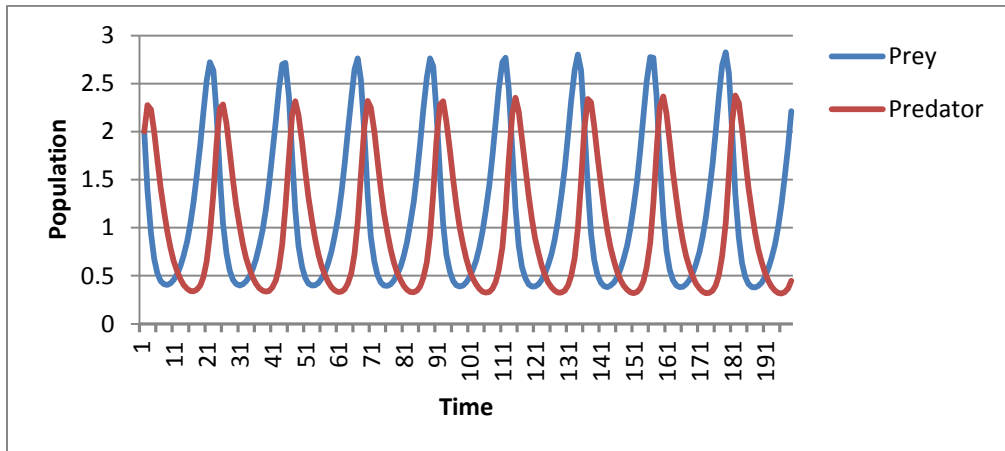


Figure 3.5: Lotka-Volterra population dynamics

3.3 The Fishing Activity Modified Lotka-Volterra Model Analysis. Once Lotka modified his model for fishing activity, his model became

$$\begin{cases} \frac{dx}{dt} = ax - bxy - ex = (a - e)x - bxy \\ \frac{dy}{dt} = -cy + dxy - ey = -(c + e)y + dxy \end{cases}$$

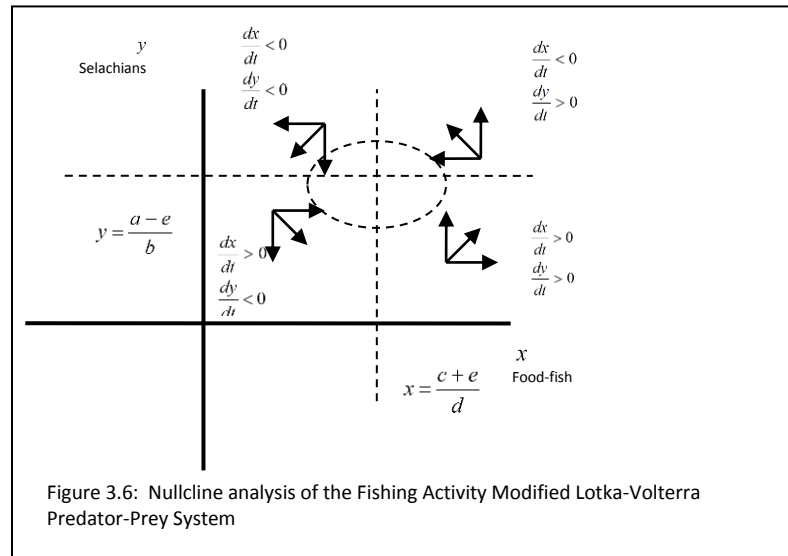
Braun and other mathematicians have analyzed this model to find equilibrium points and to draw conclusions on the inclinations of the solution curves [6].

So for prey, $\frac{dx}{dt} = 0 = (a - e)x - bxy = x(a - e - by)$ and for predators,

$\frac{dy}{dt} = 0 = -(c + e)y + dxy = -y(c + e - dx)$ produce the nullclines $x = \frac{c + e}{d}$, $y = \frac{a - e}{b}$. An

analysis of the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ leads to the conclusions drawn in Figure 3.6 below,

which are identical to the first model, provided $a - e > 0$.



When $\frac{dx}{dt} = 0 = (a - e)x - bxy = x(a - e - by)$ and

$\frac{dy}{dt} = 0 = -(c + e)y + dxy = -y(c + e - dx)$, the equilibrium points (0,0) and

$\left(\frac{c+e}{d}, \frac{a-e}{b}\right)$ are produced. The stability of each point can be determined by analyzing

the Jacobian matrix:

$$J(x, y) = \begin{bmatrix} a-e-by & -bx \\ dy & dx-c-e \end{bmatrix}$$

For the fixed point at the origin,

$$J(0,0) = \begin{bmatrix} a-e & 0 \\ 0 & -c-e \end{bmatrix}$$

producing the characteristic equation $\lambda^2 - (a-c-2e)\lambda - (a-e)(c+e) = 0$, with eigenvalues $a-e$ and $-(c+e)$. Since $a > 0$ and $c > 0$, and as long as $a-e > 0$, the eigenvalues have opposite signs, implying that $(0,0)$ is a saddle point. This unstable equilibrium point once again represents the point of extinction of both species.

For the second fixed point $\left(\frac{c+e}{d}, \frac{a-e}{b}\right)$,

$$J\left(\frac{c+e}{d}, \frac{a-e}{b}\right) = \begin{bmatrix} 0 & -\frac{b(c+e)}{d} \\ \frac{(a-e)d}{b} & 0 \end{bmatrix}$$

producing the characteristic equation $\lambda^2 + (a-e)(c+e) = 0$, with eigenvalues $\pm i\sqrt{(a-e)(c+e)}$. The eigenvalues are purely imaginary with closed trajectories once again surrounding the stable point on the phase portrait. Similarly, levels of predator-prey oscillate around the point. These conclusions can be observed in the phase

portraits of the system of differential equations in Figures 3.7 – 3.9 below. In addition, the relationships of the two equilibrium points remain the same with various values of the constants just as in the first model.

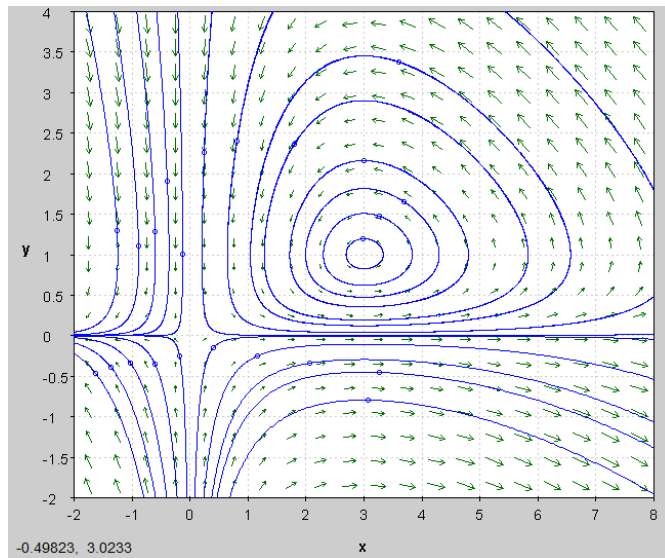


Figure 3.7: Phase portrait with $a = 10$, $b = 5$, $c = 10$, $d = 5$, $e = 5$

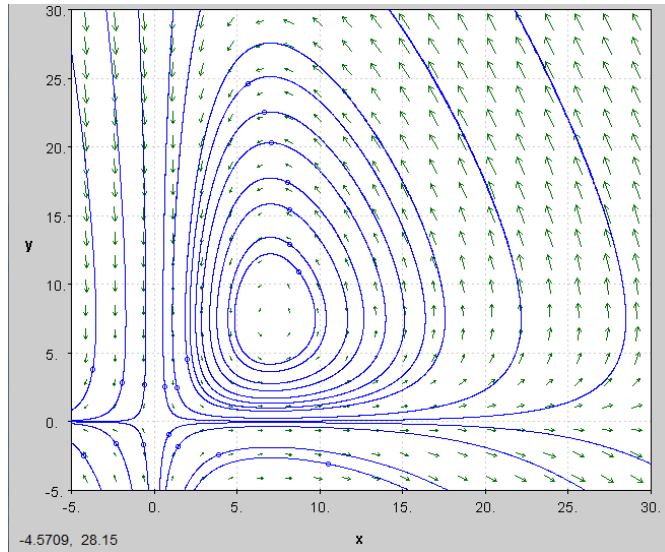


Figure 3.8: Phase portrait with $a = 20$, $b = 2$, $c = 30$, $d = 5$, $e = 5$

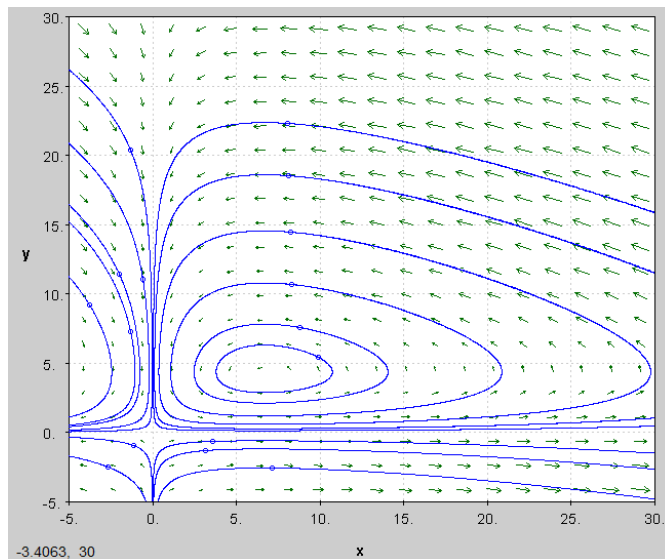


Figure 3.9: Phase portrait with $a = 40$, $b = 8$, $c = 15$, $d = 3$, $e = 5$

For the fishing activity modified Lotka-Volterra model, the data D'Ancona collected agrees with the analysis. During WWI, fishing fleets were less likely to go out.

If e , the intensity of fishing activity, is decreased, then $\frac{a-e}{b}$ increases, so selachian populations increase and $\frac{c+e}{d}$ decreases so food-fish populations decrease. Therefore, e would decrease and the equilibrium point for food fish would decrease while the equilibrium point for selachians would increase. The result of an increase in the percentage of selachians available is reflected in the data.

However, there is insufficient data to obtain more than a general overview of the effect of fishing activity on the population dynamics. Future statistical studies would need to be done to verify results for other geographical locations and extend the model to other, more complex, situations.

3.4 Holling Logistic Model Analysis. Holling did many analyses [7] on his model

$$\begin{cases} \frac{dx}{dt} = a\left(1 - \frac{x}{k}\right)x - bxy \\ \frac{dy}{dt} = -y(c - dx) \end{cases}$$

This system's equilibrium points can be found by first finding the nullclines.

For prey

$$\begin{aligned}\frac{dx}{dt} = 0 &= a\left(1 - \frac{x}{k}\right)x - bxy \\ 0 &= x\left(a\left(1 - \frac{x}{k}\right) - by\right)\end{aligned}$$

This implies that either $x = 0$ or that

$$\begin{aligned}a\left(1 - \frac{x}{k}\right) - by &= 0 \\ by &= a\left(1 - \frac{x}{k}\right) \\ y &= \frac{a}{b}\left(1 - \frac{x}{k}\right)\end{aligned}$$

So the nullclines for prey are $x = 0$ and $y = \frac{a}{b}\left(1 - \frac{x}{k}\right)$, where a and b are constants.

For predators $\frac{dy}{dt} = 0$ so $0 = -y(c - dx)$. Since c and d are constants, the zero nullcline

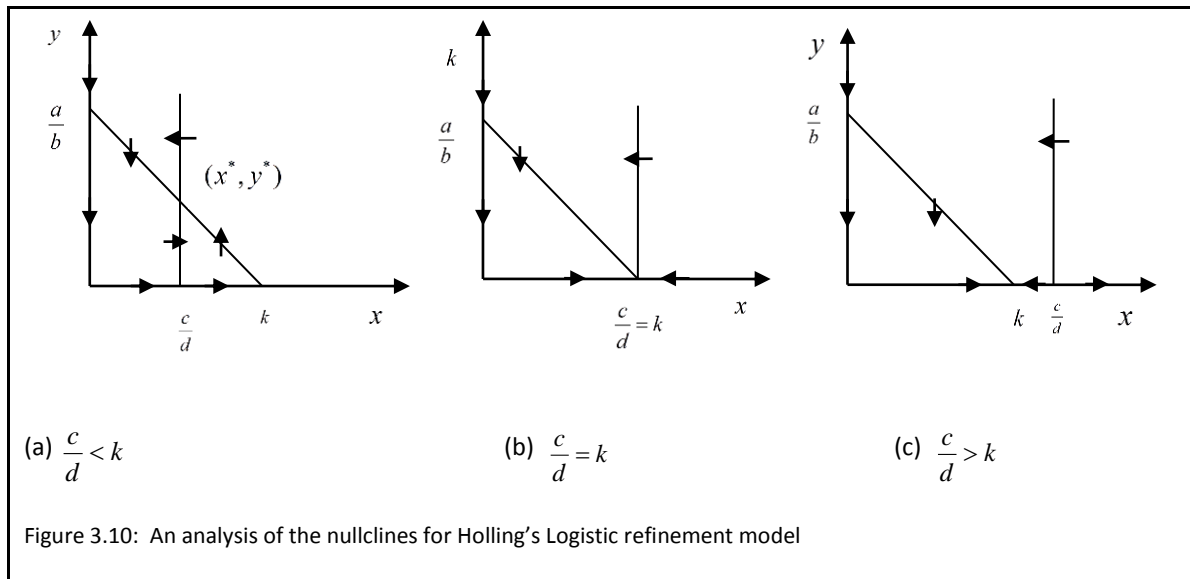
for predators is a constant defined by prey numbers, $x = \frac{c}{d}$.

An analysis of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ leads to Figure 3.10 below. In part a, $\frac{c}{d} < k$, which

gives the equilibrium points $(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b} - \frac{ac}{kbd}\right)$, $(0, 0)$, $\left(\frac{c}{d}, 0\right)$, and $(k, 0)$. In part b,

$\frac{c}{d} = k$ which gives the equilibrium point $(0,0)$ and $(k,0)$. In part c, $\frac{c}{d} > k$ yields

equilibrium points $(0,0)$, $(k,0)$ and $\left(\frac{c}{d}, 0\right)$.



The Jacobian matrix is given by the partial derivatives of the system:

$$J(x, y) = \begin{bmatrix} a - \frac{2ax}{k} - by & -bx \\ dy & dx - c \end{bmatrix}$$

The linearization at the interesting equilibrium point is

$$J(x^*, y^*) = \begin{bmatrix} \frac{-ac}{kd} & \frac{-bc}{d} \\ \frac{adk - ac}{bk} & 0 \end{bmatrix}$$

The trace of the matrix is $-\frac{ac}{kd}$ which must be negative since a, c, k and d are

positive parameters. The determinant $\frac{ac(dk - c)}{dk}$ is positive, the eigenvalues are

negative, and the equilibrium point is a sink as long as $dk - c > 0$.

The equilibrium point $\left(\frac{c}{d}, \frac{a}{b} - \frac{ac}{kbd}\right) = \left(\frac{c}{d}, \frac{a}{b}\left(1 - \frac{c}{kd}\right)\right)$ determines that if kd is

smaller than c the predator population would be negative. Since the population cannot be negative, this implies $dk - c > 0$. Therefore, as long as the equilibrium point remains above the x -axis, it will be a sink, as shown in Figure 3.11, the phase portrait of the Holling Logistic Model.

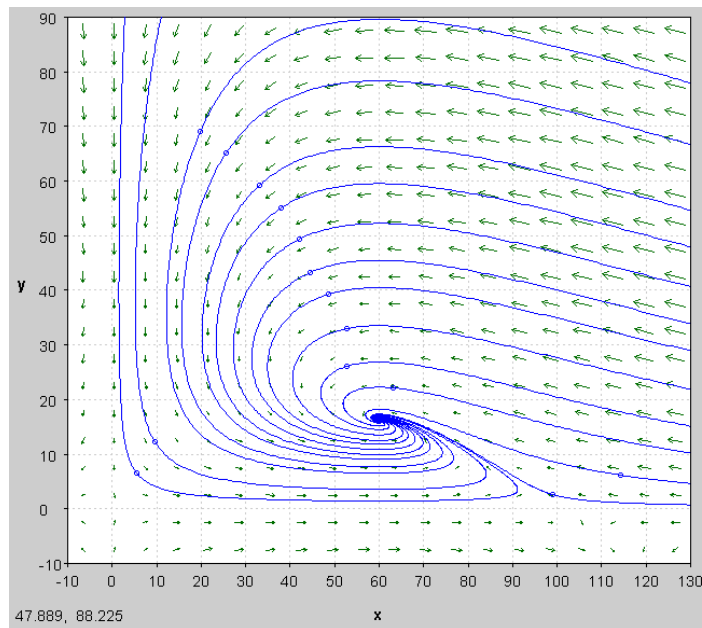


Figure 3.11: Phase Portrait of Holling Logistic Model, where $a = 0.4$,

$b = 0.1, c = 0.3, d = 0.005, k = 100$

The solutions to this system are very different from the Lotka-Volterra system. In the Lotka Volterra system, solutions cycle around the equilibrium point of interest. The solution curves in the Holling Logistic model show a much safer cycling. In the Lotka Volterra system, any agitations (e.g. an accidental death to a normally insignificant portion of prey population) in the predator or prey system could push the populations down to 0 and lead to extinction. In this modified system, after a maximum of one pass close the axis, the solutions spiral away from the possible extinction and toward the equilibrium point of interest.

Safe, stable populations seem to be guaranteed as time approaches infinity if the equilibrium point is a sufficient distance from both axes to protect from random distresses. With predator actions in mind, namely a predator will have a limited capacity to kill and a limited motivation to kill once they have enough sustenance, a weakness is perceived with this system. Utilizing the prey equation, the harm done to the prey population is bx per predator. Therefore, in a given time interval, each predator will slay some fixed fraction of the prey population. This linear predation is not a realistic interpretation of predator-prey interactions.

So although this model is an improvement, it still does not model the predator-prey interactions perfectly. A better model would be of the form

$$\begin{cases} \frac{dx_1}{dt} = x_1(t) \cdot f(x_1(t), x_2(t)) \\ \frac{dx_2}{dt} = x_2(t) \cdot g(x_1(t), x_2(t)) \end{cases}$$

where $f(x_1(t), x_2(t))$ and $g(x_1(t), x_2(t))$ model the prey and predator situation in a logistic fashion. The next model will consider this approach to determine if the model is a better fit for predator-prey interactions.

3.5 Holling-Tanner Model Analysis. Several dynamical behavior analyses of the Holling-Tanner model

$$\begin{aligned} \frac{dx}{dt} &= a \left(1 - \frac{x}{k} \right) x - yw \frac{x}{r+x} \\ \frac{dy}{dt} &= c \left(1 - \frac{y}{x} \right) y \end{aligned}$$

have been studied extensively, by Tanner [8] and May[9] among others, researching the model analysis and trends of solution curves.

The system's equilibrium points are found as before by starting with the nullclines. For prey

$$\begin{aligned} \frac{dx}{dt} = 0 &= a \left(1 - \frac{x}{k} \right) x - yw \frac{x}{r+x} \\ 0 &= x \left(a \left(1 - \frac{x}{k} \right) - \frac{yw}{r+x} \right) \end{aligned}$$

Either $x = 0$ or $0 = a \left(1 - \frac{x}{k} \right) - \frac{yw}{r+x}$, so

$$\begin{aligned}
0 &= a\left(1 - \frac{x}{k}\right) - \frac{yw}{r+x} \\
\frac{yw}{r+x} &= a\left(1 - \frac{x}{k}\right) \\
yw &= a\left(1 - \frac{x}{k}\right)(r+x) \\
y &= \frac{a}{w}\left(1 - \frac{x}{k}\right)(r+x)
\end{aligned}$$

Hence, the nullclines are $x=0$ and $y = \frac{a}{w}\left(1 - \frac{x}{k}\right)(r+x)$ for prey. Manipulating the

second nullcline gives intercepts of k and $-r$ and $y = -\frac{a}{wk}x^2 + \frac{ak-ar}{wk}x + \frac{ar}{w}$. Since

all parameters are positive, it can be concluded that the parabolic nullcline opens down.

Some values of k and r would have the vertex in the first quadrant.

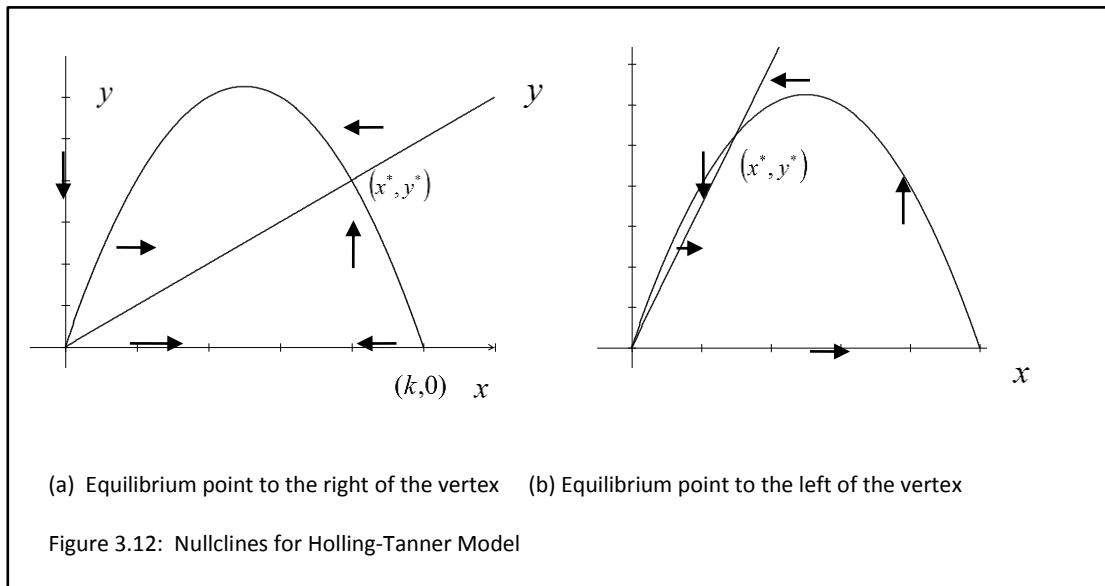
The predator nullclines are found similarly using

$$\frac{dy}{dt} = 0 = c\left(1 - \frac{y}{x}\right)y \text{ so either } y=0 \text{ or } c\left(1 - \frac{y}{x}\right) = 0.$$

$$\begin{aligned}
c\left(1 - \frac{y}{x}\right) &= 0 \\
1 - \frac{y}{x} &= 0 \\
1 &= \frac{y}{x} \\
y &= \frac{x}{j}
\end{aligned}$$

The predator nullclines are $y=0$ and $y = \frac{x}{j}$.

The equilibrium points can be found from the intersection of the nullclines as shown in figure 3.12. There are two equilibrium points observed: one at $(k,0)$ and another at (x^*, y^*) where the nontrivial prey nullcline intersects the nontrivial predator nullcline in the first quadrant. This equilibrium point could be located right of the vertex of the parabolic nullcline as shown in part a, or left of the vertex as shown in part b.



While the exact coordinates can be found by setting the equations for the nullclines equal and solving, this does not provide useful information because of the large number of parameters. Instead, if x , y , r , and k are divided by the values at the critical points, namely $x^* = 1$ and $y^* = j^{-1}$, following May's procedure [8], [9], the Jacobian becomes

$$J(x^*, y^*) = \begin{bmatrix} a\left(-\frac{1}{k} + \frac{w}{(aj)(1+r)^2}\right) & -\frac{w}{1+r} \\ \frac{c}{j} & -c \end{bmatrix}$$

The determinant of this matrix is

$$\begin{aligned} & -ac\left(-\frac{1}{k} + \frac{w}{(aj)(1+r)^2}\right) + \frac{cw}{j(1+r)} \\ &= \frac{ac}{k} - \frac{acw}{(aj)(1+r)^2} + \frac{cw}{j(1+r)} \\ &= ac\left(\frac{1}{k} + \frac{w + wr - w}{aj(1+r)^2}\right) \\ &= ac\left(\frac{1}{k} + \frac{wr}{aj(1+r)^2}\right) \end{aligned}$$

All parameters are positive so the determinant must be positive.

If the trace is less than zero, the eigenvalues will be negative which will make the equilibrium point a sink.

$$\begin{aligned} & a\left(-\frac{1}{k} + \frac{w}{(aj)(1+r)^2}\right) - c < 0 \\ & -\frac{1}{k} + \frac{w}{(aj)(1+r)^2} < \frac{c}{a} \end{aligned}$$

For $x^* = 1$ and $y^* = \frac{1}{j}$ for our equilibrium point and the nullcline $y = \frac{a}{w}\left(1 - \frac{x}{k}\right)(r+x)$,

$$\frac{w}{aj} = \left(1 - \frac{1}{k}\right)(1+r) \text{ so}$$

$$\begin{aligned}
& -\frac{1}{k} + \left(1 - \frac{1}{k}\right)(1+r) \frac{1}{(1+r)^2} < \frac{c}{a} \\
& -\frac{1}{k} + \frac{1 - \cancel{1/k}}{1+r} < \frac{c}{a} \\
& \frac{k-r-2}{k(1+r)} < \frac{c}{a} \\
& \frac{2\left(\frac{k-r}{2} - 1\right)}{k(1+r)} < \frac{c}{a}
\end{aligned}$$

For our equilibrium point, the trace of the characteristic equation will be greater than

zero if $\frac{2\left(\frac{k-r}{2} - 1\right)}{k(1+r)} < \frac{c}{a}$. The maximum of the prey nullcline is at $\frac{k-r}{2}$. If the x -value

of the maximum of the nullcline is less than the x -value of the equilibrium point, that is

$\frac{k-r}{2} < (1 = x^*)$, then $\frac{k-r}{2}$ and $\frac{2\left(\frac{k-r}{2} - 1\right)}{k(1+r)}$ will be negative. Therefore, for all values

of $\frac{c}{a}$ the model is stable if the equilibrium point is to the right of the vertex of the

parabolic nullcline, as shown in figure 3.13.

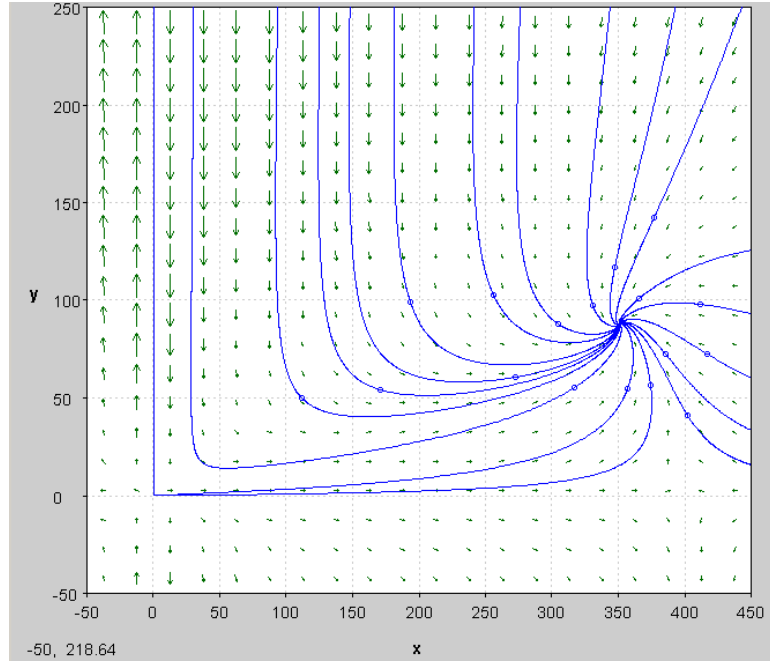


Figure 3.13: Phase portrait of Holling-Tanner Model with equilibrium point to the right of the vertex,

where $a=1$, $c=1$, $j=4$, $k=400$, $r=5$, $w=0.5$

If x^* is to the left of the parabolic prey nullcline, that is $\frac{k-r}{2} > (1 = x^*)$ then

both fractions are positive. The equilibrium point will be stable for $\frac{a}{c}$ larger than

$$\frac{2\left(\frac{k-r}{2} - 1\right)}{k(1+r)}, \text{ as shown in figure 3.14.}$$

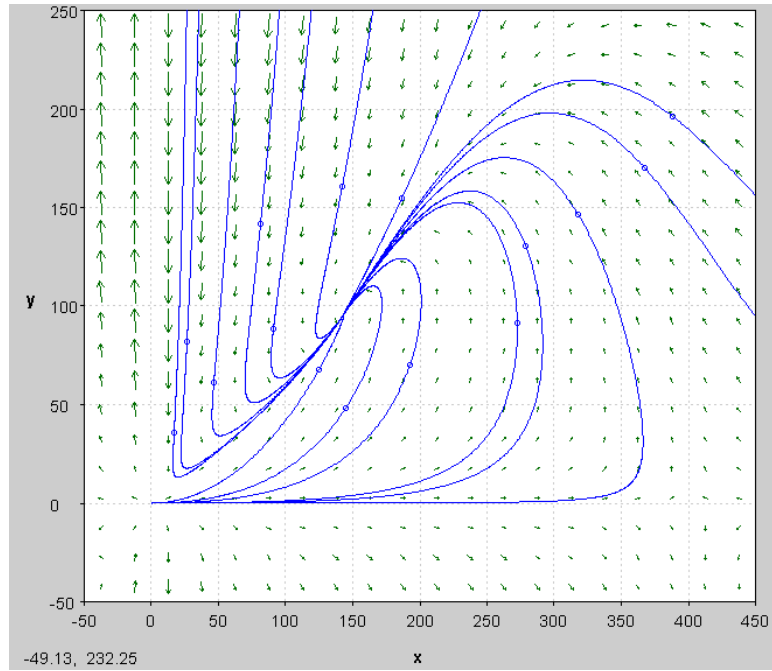


Figure 3.14: Phase portrait of Holling-Tanner Model with stable equilibrium point to the left of the vertex, where

$$a = 0.5, c = 1, j = 1.5, k = 400, r = 5, w = 0.5$$

Otherwise, solutions started near the equilibrium point will travel away from the equilibrium point and out toward a limit cycle, as shown in figure 3.15.

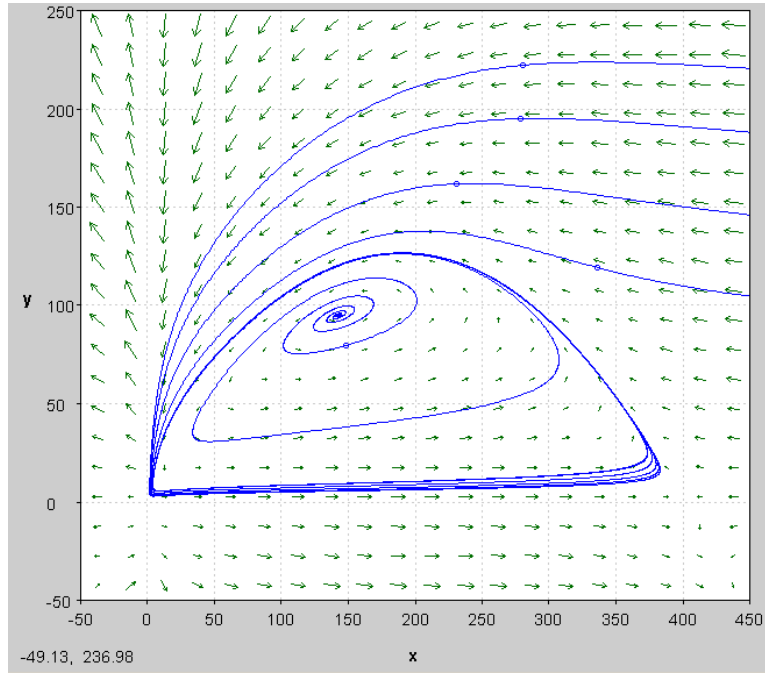


Figure 3.15: Phase portrait of Holling-Tanner Model with unstable equilibrium point to the left of the vertex, where

$$a = 0.5, c = 0.1, j = 1.5, k = 400, r = 5, w = 0.5$$

Theoretically, different behaviors can be exhibited and manipulated depending on the parameters from the Holling-Tanner Model. Conversely, real world behavior from predator-prey communities is more difficult. For instance, changing a or c could mean interfering with the natural reproductive rates of one species while leaving the other alone. In addition, lowering the nutritional value of prey could increase j the satiety of a predator which could make predators kill more often, but this manipulation could possibly affect health and longevity.

Furthermore, this system is not linear so the process of linearization does hold for solutions near the equilibrium point, but there is no guarantee that these solutions

will behave similarly further away. Solutions could possibly start near the unstable equilibrium point and do many different things further out. Further analysis is necessary to fully understand the long term behavior of this system.

Currently, the Holling-Tanner model describes the interactions of the predator-prey communities much better than other models. However, it does not consider human intervention, such as seasonal hunting or fishing. Obviously, human involvement would change the dynamics of the model and the equilibrium. For example, fishing can destroy an ecosystem by reducing the population to a level which would not recover and lead to extinction. Mankind can equally choose to protect such ecosystems by interventions, such as establishing criteria and resolutions for hunting and fishing through quotas or restrained time periods of hunting or fishing. For instance, consider a fishing circumstance where predator and prey are caught and removed from the ecosystem. How much fishing, i.e. how many fish can be caught, in order to maximize the population of both the predator and prey populations? In order to solve such problems, an optimal control problem must be stated. This topic will be considered in the last section of this thesis.

CHAPTER IV

REVIEW OF OPTIMAL CONTROL THEORY & LITERATURE

Optimal control theory, principally due to the work of Lev Pontryagin and his collaborators in the Soviet Union and Richard Bellman in the United States, is a branch of mathematics developed to find optimal ways to control a dynamic system, or a system that evolves over time. A clear mathematical description of the system to be optimized, constraints imposed on the system, and the objective function to be maximized (or minimized) is required before any solution can be attempted.

For instance, consider a simple case where a state equation,

$$\dot{x} = f(x(t), u(t)), x(0) = x_0$$

where $x(t)$ is a state variable that depends on time and $u(t)$ is a control function to achieve the desired criterion, and x_0 is the initial state, is introduced that describes the behavior of the underlying dynamical system. Next, constraints are imposed on state and control variables. For instance, for each $t \in [0, T]$, $x(t)$ and $u(t)$, could satisfy

$$g(x(t), u(t)) \geq 0, t \in [0, T].$$

The objective function

$$J(u) = \int_{t_0}^{t_1} F(t, x(t), u(t)) dt \rightarrow \max_{u(\cdot) \in D(T)}$$

gives and quantitative measure of the performance of the system over time, with appropriate measure of quantities such as profits, sales, or populations.

The goal is to find an admissible control, $u^*(t)$, which maximizes (or minimizes) the objective function subject to the state equation and control restraints. Pontryagin's Maximum Principal (PMP) states that if $(x^*(t), u^*(t))$ is an optimal pair, then there exists a continuous function, $\lambda(t)$, the adjoint function, such that for all t in $[t_0, t_1]$, $u = u^*(t)$ maximizes the so-called Hamiltonian, defined as $H(x(t), u, \lambda(t), t)$ for $u(\cdot) \in D(T)$, where $D(T)$ is the set of all permissible controls. In addition, the PMP specifies that

$$H(x^*(t), u^*(t), \lambda^*(t), t) \leq H(x^*(t), u(t), \lambda^*(t)) \text{ for all } u \in D(T)$$

The PMP's conditions are a set of necessary, but not sufficient conditions for the optimality. Further information on the introduction of optimal control theory can be found in many books, such as Kirk [11].

Since the development of optimal control theory, many mathematicians have studied and expanded work in this field. For instance, Berkovitz sketches out the historical development of optimal control theory in 1976 [12]. In this paper, he outlines problems, such as the servo problem, the linear time-optimal problem, nonlinear optimal control problem, calculus of variations and sub problems derived, and the maximum principle.

Additionally, Hartl, Sethi, and Vickson compile an overview of the various forms of Pontryagin's maximum principle and the different sets of optimality conditions, with

“State Variable Inequality Constraints (SVICs)” [13]. In this work, the authors not only demonstrate the different forms of analysis but also concentrate on applications that frequently appear in the engineering and mechanics fields.

Some mathematicians are focusing specific methods of solutions using analytical techniques. For instance, Rupp uses the technique of multipliers applied to a nonlinear optimal control problem [14]. In this work, Rupp outlines the technique by saying:

“Hestenes’ method of multipliers is to embed the differentially constrained problem in a family of unconstrained problems so as to preserve standard sufficiency criteria. Given an initial estimate of the Lagrange multipliers, a convergent sequence of arcs is generated. They are minimizing with respect to members of the above family, and their limit is the solution to the differentially constrained problem.”

Rupp then reviews the sufficiency criteria and sufficiency theorems in order to apply the method. Finally, he applies the method to a nonlinear optimal control problem.

An optimal control problem is much more difficult to solve with the existence of a singular control. This degree of challenge means that the same methods cannot be used with all optimal control problems. Consequently, numerical methods of solving optimal control problems have a definite place in the field. Khmelnitsky discusses a graph based numerical method for the solution of bang-bang and singular control problems [15]. Khmelnitsky begins his process by using the bounded control to find the locally optimal trajectories. Nodes and arcs are then constructed along the trajectories, although a singular arc may not be produced. The globally optimal solutions may be

found by finding the shortest path between the arcs in the graph. This numerical method can be used for finding both regular and singular optimality. Other numerical methods include using spreadsheets to solve continuous time optimal control problems, such as Nævdal's work [16]. In his work he outlines his method and illustrates several examples ranging from simple to hard. Although this method is useful, it is typically used only for problems with one variable and one control.

The technique of solving problems by using a switching function, analyzing the behavior of the function under each condition, has been utilized as well. Some work has been done by d'Onofrio, et al, while researching cancer to analyze drugs for tumor eradication [17]. Theoretical methods that allow to use switching functions for finding the type of optimal control analytically have been developed in the work of E. V. Grigorieva and E. N. Khailov (students of Lev Pontryagin) . These methods found applications in solving practical problems in economics, managements, life sciences and biology [18] – [22]. Moreover, their approach can be used for complex models with several bounded controls and several state variables. This method will be utilized in the following section.

CHAPTER V

OPTIMAL CONTROL OF HOLLING TANNER MODEL

5.1 Literature on Optimal Control of Holling-Tanner Model. The Holling-Tanner model and other predator-prey models have been the subject of mathematicians in the area of optimal control previously. Kar considered a control problem with the Holling model, altered to incorporate a prey refuge, in which two species are harvested independently [23]. Kar analyzed the model for stable limit cycles and studied a control problem with an objective function to maximize the present value of a continuous time-stream of revenues. The goal was to determine the optimal trade-off between current and future harvests in commercial fisheries. Furthermore, Luk'yanova investigated a control problem for the Holling-Tanner model with controlled parameters applied to a technology predator-prey application [24]. Old-generation technology is considered prey while the new-generation is the predator. The author utilized numerical integration techniques to make observations about the trajectories of the solution. Additional work has been done with the Holling model as well by Apreutesei [25] and Zhijun[26].

5.2 Holling-Tanner Model with Controlled Parameters. In recent literature, a modified form of the Holling-Tanner model has been presented. As referenced and described by Maiti [27], the functional response of the predator, $\frac{wxy}{r+x}$, has been replaced by $\frac{wxy}{ry+x}$.

Maiti mentions that one reason for such a switch would be to achieve the ratio dependent functional response for predators in comparison to the prey-dependent counterpart. Maiti studies this model's dynamical behavior when the environment is assumed to be deterministic and presents his results using computer simulations.

This modified Holling-Tanner model will be considered with the introduction of controls $u(t)$ and $v(t)$ that represent the possibility of harvesting or capturing the populations of predator and prey:

$$\begin{aligned}\frac{dx}{d\tau} &= a\left(1 - \frac{x}{k}\right)x - \frac{wxy}{ry+x} - ux \\ \frac{dy}{d\tau} &= c\left(1 - \frac{y}{x}\right)y - vy \\ u_{\min} &\leq u(t) \leq u_{\max}, v_{\min} \leq v(t) \leq v_{\max} \\ x(0) &= x_0 > 0, y(0) = y_0 > 0\end{aligned}$$

where $a, k, w, r, c, j > 0$ and the total population is represented by $\alpha \cdot x(T) + \beta \cdot y(T)$.

Luk'yanova studies a similar optimal control of the original version of the Holling-Tanner model [24]. In her paper, she presents and proves that the solution of her system satisfies inequalities $x_1(t) > 0$, $x_2(t) > 0$, and $t \geq 0$. Also, she theorizes that the optimal control exists for every initial state $x_1^0 > 0$ and $x_2^0 > 0$. Similarly, she writes a

proof for her theory. She determines the Hamiltonian of her system and presents an adjoint system where she offers a theory and brief proof that there are no singular cases. In this thesis, similar findings will be examined for the modified Holling-Tanner model.

The number of parameters makes an analysis challenging, so the system will be simplified using the same process as used in as Maiti's work by means of substitutions:

$$x_1 = \frac{x}{k}, \quad x_2 = \frac{ry}{k}, \quad \text{and} \quad t = a\tau.$$

For the first equation:

$$\begin{aligned} \frac{dx}{d\tau} \cdot \frac{dx_1}{dx} \cdot \frac{d\tau}{dt} &= \left(a \left(1 - \frac{x}{k} \right) x - \frac{wxy}{ry+x} - ux \right) \left(\frac{1}{k} \right) \left(\frac{1}{a} \right) \\ \frac{dx_1}{dt} &= x_1 \left(1 - x_1 \right) - \frac{w}{ka} \cdot \frac{kx_1 \cdot \cancel{kx_2} / r}{kx_2 + kx_1} - \frac{ux}{ka} \\ \frac{dx_1}{dt} &= x_1 \left(1 - x_1 \right) - \frac{w}{ra} \cdot \frac{x_1 x_2}{x_2 + x_1} - \frac{ux}{ka} \end{aligned}$$

And then the second equation:

$$\begin{aligned} \frac{dy}{d\tau} \cdot \frac{dx_2}{dy} \cdot \frac{d\tau}{dt} &= \left(c \left(1 - \frac{jy}{x} \right) y - vy \right) \left(\frac{r}{k} \right) \left(\frac{1}{a} \right) \\ \frac{dx_2}{dt} &= \frac{cx_2}{a} \left(1 - \frac{j}{r} \cdot \frac{x_2}{x_1} \right) - \frac{rvy}{ka} \end{aligned}$$

The system becomes

$$\begin{cases} \frac{dx_1}{dt} = (1-x_1)x_1 - \frac{w}{ra} \cdot \frac{x_1x_2}{x_1+x_2} - u_1x_1 \\ \frac{dx_2}{dt} = \frac{c}{a} \left(1 - \frac{j}{r} \cdot \frac{x_2}{x_1} \right) x_2 - u_2x_2 \end{cases}$$

The notation $\beta = \frac{w}{ra}$, $\alpha = \frac{c}{a}$, and $\gamma = \frac{j}{r}$ will be introduced to further simplify.

For the first equation:

$$\frac{dx_1}{dt} = (1-x_1)x_1 - \beta \cdot \frac{x_1x_2}{x_1+x_2} - u_1x_1$$

And the second equation:

$$\frac{dx_2}{dt} = \alpha \left(1 - \gamma \cdot \frac{x_2}{x_1} \right) x_2 - u_2x_2$$

Finally, the system becomes

$$\begin{cases} \frac{dx_1}{dt} = (1-x_1)x_1 - \frac{\beta x_1x_2}{x_1+x_2} - u_1x_1 \\ \frac{dx_2}{dt} = \alpha \left(1 - \gamma \frac{x_2}{x_1} \right) x_2 - u_2x_2 \\ x_1(0) = x_1^0 > 0 \\ x_2(0) = x_2^0 > 0 \end{cases} \quad (1)$$

where α, β, γ are given positive constants.

The following Lemma and proofs illustrate an important property of the system

(1) that will be utilized in the optimal control situation.

Lemma 1. *Solutions of system (1) are positive for all $t \in [0, T]$.*

Proof 1a. Using the first equation of system (1)

$$\frac{dx_1}{dt} = (1 - x_1)x_1 - \frac{\beta x_1 x_2}{x_1 + x_2} - u_1 x_1$$

and integrating after separation of variables

$$\begin{aligned} \frac{dx_1}{dt} &= (1 - x_1)x_1 - \frac{\beta x_1 x_2}{x_1 + x_2} - u_1 x_1 \\ \frac{dx_1}{x_1} &= (1 - x_1) - \frac{\beta x_1 x_2}{x_1 + x_2} - u_1 \\ \int \frac{dx_1}{x_1} &= \left(\int (1 - x_1) - \frac{\beta x_1 x_2}{x_1 + x_2} - u_1 \right) dt + C \\ \ln|x_1| &= \int \left((1 - x_1) - \frac{\beta x_1 x_2}{x_1 + x_2} - u_1 \right) dt + C \\ x_1 &= e^{\int \left((1 - x_1) - \frac{\beta x_1 x_2}{x_1 + x_2} - u_1 \right) dt + C} \end{aligned}$$

produces

$$x_1(t) = x_1^0 e^{\int_0^t \left((1 - x_1(\xi)) - \frac{\beta x_1(\xi) x_2(\xi)}{x_1(\xi) + x_2(\xi)} - u_1(\xi) \right) d\xi} \quad (2)$$

Therefore, since $x_1^0 > 0$ and $e^{\int_0^t \left((1 - x_1(\xi)) - \frac{\beta x_1(\xi) x_2(\xi)}{x_1(\xi) + x_2(\xi)} - u_1(\xi) \right) d\xi} > 0$ then $x_1(t) > 0$, and the

lemma is proven for the first equation of system (1).

Proof 1b. Similarly, using the second equation of system (1)

$$\frac{dx_2}{dt} = \left(\alpha - \gamma \frac{x_2}{x_1} \right) x_2 - u_2 x_2$$

and integrating after separation of variables

$$\begin{aligned}
\frac{dx_2}{dt} &= \left(\alpha - \gamma \frac{x_2}{x_1} \right) x_2 - u_2 x_2 \\
\frac{dx_2}{x_2} &= \left(\alpha - \gamma \frac{x_2}{x_1} \right) - u_2 \\
\int \frac{dx_2}{x_2} &= \int \left(\left(\alpha - \gamma \frac{x_2}{x_1} \right) - u_2 \right) dt + C \\
\ln|x_2| &= \int \left(\left(\alpha - \gamma \frac{x_2}{x_1} \right) - u_2 \right) dt + C \\
x_2 &= e^{\int \left(\left(\alpha - \gamma \frac{x_2}{x_1} \right) - u_2 \right) dt + C}
\end{aligned}$$

produces

$$x_2(t) = x_2^0 e^{\int_0^t \left(\left(\alpha - \gamma \frac{x_2(\xi)}{x_1(\xi)} \right) - u_2(\xi) \right) d\xi} \quad (3)$$

Therefore, since $x_2^0 > 0$ and $e^{\int_0^t \left(\left(\alpha - \gamma \frac{x_2(\xi)}{x_1(\xi)} \right) - u_2(\xi) \right) d\xi} > 0$ then $x_2(t) > 0$, and the lemma is proven for the second equation of system (1).

This property can be generalized for predator-prey systems since almost all utilize the form

$$\begin{cases} \frac{dx_1}{dt} = x_1(t) \cdot f(x_1(t), x_2(t)) \\ \frac{dx_2}{dt} = x_2(t) \cdot g(x_1(t), x_2(t)) \end{cases}$$

For the first system, integrating after separation of variables

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(t) \cdot f(x_1(t), x_2(t)) \\ \frac{dx_1}{x_1} &= f(x_1(t), x_2(t)) \\ \int \frac{dx_1}{x_1} &= \int (f(x_1(t), x_2(t))) dt + C \\ \ln|x_1| &= \int (f(x_1(t), x_2(t))) dt + C \\ x_1 &= e^{\int (f(x_1(t), x_2(t))) dt + C}\end{aligned}$$

produces $x_1 = x_1^0 e^{\int (f(x_1(t), x_2(t))) dt}$. Using separation of variables for the second system

likewise produces $x_2 = x_2^0 e^{\int (g(x_1(t), x_2(t))) dt}$. With the initial conditions $x_1^0 > 0$ and $x_2^0 > 0$

then for all $f(x_1(t), x_2(t)), g(x_1(t), x_2(t))$, $x_1(t) > 0$ and $x_2(t) > 0$. This observation is interesting in that it appears to apply to most predator-prey systems that have been utilized to model scenarios.

Lemma 1 illustrates the property that the solutions $x_1(t)$ and $x_2(t)$ must be positive. Utilizing this property of positiveness of solutions and equation (2), the following inequality can be written.

$$x_1(t) < x_1^0 e^{\int_0^t (1-u_1(\xi)) d\xi} \leq x_1^0 e^{(1-u_{\min}^1)T}$$

In addition, from equation (3) and the property from Lemma 1 of the positiveness of solutions $x_1(t)$ and $x_2(t)$, a second inequality about the predator-prey system can be written as follows:

$$x_2(t) < x_2^0 e^{\int_0^t (\alpha - u_2(\xi)) d\xi} \leq x_2^0 e^{(\alpha - u_{\min}^2)T}$$

Then, via the work of Hartman on differential equations [28], it follows that $x_1(t)$ and $x_2(t)$ of equation (1) exist on $(0, T)$ and obey the inequalities:

$$x_1(t) < x_1^0 e^{\int_0^t (1 - u_1(\xi)) d\xi} \leq x_1^0 e^{(1 - u_{\min}^1)T}, \quad x_2(t) < x_2^0 e^{\int_0^t (\alpha - u_2(\xi)) d\xi} \leq x_2^0 e^{(\alpha - u_{\min}^2)T}, \quad t \in (0, T).$$

5.3 Statement of Optimal Control Problem. Fisheries management utilizes fisheries science in order to obtain maximum sustainable yield from a target species. Modern fisheries management is typically idealized by rules constructed by defined objectives and management to implement the rules through monitoring and surveillance. For instance, quotas on amounts of catch of fish are stated and management must make sure that fishing does not exceed this quota.

For this instance, consider the optimal control problem of maximizing the entire population at the terminal time T under harvesting efforts

$$0 \leq u_{\min}^{(1)} \leq u_1(t) \leq u_{\max}^{(1)}, 0 \leq u_{\min}^{(2)} \leq u_2(t) \leq u_{\max}^{(2)} :$$

$$J(u_1, u_2) = Ax_1(T) + Bx_2(T) \rightarrow \max_{u_1(\cdot), u_2(\cdot) \in D(t)} \quad (4)$$

Here A and B are nonnegative constants, with both constants less than or equal to one, so called scaling factors. These constants can be interpreted as the economic importance of certain species. Thus, for $B = 1$ then the predator population is important, for $B = 0.5$ the predator population it is less important, and if $B = 0$, it is not that important. The existence of the optimal solution to a problem of optimal control of equations (1) and (4) follows from the work of Lee and Markus on optimal control theory [29].

Applying the Pontryagin Maximum Principle, the Hamiltonian for this optimal control problem is as follows:

$$H = \left((1 - x_1)x_1 - \frac{\beta x_1 x_2}{x_1 + x_2} - u_1 x_1 \right) \psi_1 + \left(x_2 \left(\alpha - \gamma \cdot \frac{x_2}{x_1} \right) - u_2 x_2 \right) \psi_2$$

To find our optimal solution, there must exist a non-trivial solution $\psi^*(t) = (\psi_1^*(t), \psi_2^*(t))$

of the adjoint system can be obtained using $\dot{\psi}_1(t) = \frac{-\partial H}{\partial x_1}$ and $\dot{\psi}_2(t) = \frac{-\partial H}{\partial x_2}$. The first

equation from the predator-prey system (1) can be used to find $\dot{\psi}_1(t)$ as follows.

$$\begin{aligned} \dot{\psi}_1 &= \frac{-\partial H}{\partial x_1} = - \left(\left(1 - 2x_1 - \frac{(x_1 + x_2)(\beta x_2) - \beta x_1 x_2}{(x_1 + x_2)^2} - u_1 \right) \psi_1 + \gamma \left(\frac{x_2}{x_1} \right)^2 \psi_2 \right) \\ \dot{\psi}_2 &= - \left(1 - 2x_1 - \beta \cdot \left(\frac{x_2}{x_1 + x_2} \right)^2 - u_1 \right) \psi_1 - \gamma \left(\frac{x_2}{x_1} \right)^2 \psi_2 \end{aligned}$$

Likewise, the second equation from system (1) can be utilized to find $\dot{\psi}_2(t)$.

$$\begin{aligned}\dot{\psi}_2 &= \frac{-\partial H}{\partial x_2} = -\left(\left(-\frac{(x_1+x_2)(\beta x_1)-(\beta x_1 x_2)}{(x_1+x_2)^2}\right)\psi_1 + \left(\left(\alpha - 2\gamma \cdot \frac{x_2}{x_1}\right) - u_2\right)\psi_2\right) \\ \dot{\psi}_2 &= \beta \cdot \left(\frac{x_1}{x_1+x_2}\right)^2 \psi_1 - \left(\alpha - 2\gamma \cdot \frac{x_2}{x_1} - u_2\right)\psi_2\end{aligned}$$

Finally, these derivations yield the adjoint system as listed below.

$$\begin{cases} \dot{\psi}_1 = -\left(1 - 2x_1 - \beta \cdot \left(\frac{x_2}{x_1+x_2}\right)^2 - u_1\right)\psi_1 - \gamma \left(\frac{x_2}{x_1}\right)^2 \psi_2 \\ \dot{\psi}_2 = \beta \cdot \left(\frac{x_1}{x_1+x_2}\right)^2 \psi_1 - \left(\alpha - 2\gamma \cdot \frac{x_2}{x_1} - u_2\right)\psi_2 \end{cases}$$

with the boundary condition $\psi_1(T) = A$ and $\psi_2(T) = B$. Since H is linear for controls,

$u_1(t)$ and $u_2(t)$, it follows from PMP that optimal controls $u_1^*(t)$ and $u_2^*(t)$ can be

written as

$$u_1^*(t) = \begin{cases} u_{\max}^1, & \text{if } L_1(t) > 0 \\ [u_{\min}^1, u_{\max}^1], & \text{if } L_1(t) = 0 \\ u_{\min}^1, & \text{if } L_1(t) < 0 \end{cases}$$

and

$$u_2^*(t) = \begin{cases} u_{\max}^2, & \text{if } L_2(t) > 0 \\ [u_{\min}^2, u_{\max}^2], & \text{if } L_2(t) = 0 \\ u_{\min}^2, & \text{if } L_2(t) < 0 \end{cases}$$

Here $L_1(t) = -x_1\psi_1$, $L_2(t) = -x_2\psi_2$ are so-called functions of switching because their

behavior determines the type of the optimal control. Then, following to the ideas of

Grigorieva and Khailov [18] – [22], an attempt will be made to rewrite the adjoint system in terms of the new variables $L_1(t)$ and $L_2(t)$.

Making the change of variables using $L_1 = -x_1\psi_1$, then

$$\begin{aligned}\dot{L}_1 &= -\dot{x}_1\psi_1 - x_1\dot{\psi}_1 \\ \dot{L}_1 &= \left(-(1-x_1)x_1 + \frac{\beta x_1 x_2}{x_1 + x_2} + u_1 x_1 \right) \psi_1 + x_1 \left(1 - 2x_1 - \beta \cdot \left(\frac{x_2}{x_1 + x_2} \right)^2 - u_1 \right) \psi_1 + \gamma \cdot x_1 \cdot \left(\frac{x_2}{x_1} \right)^2 \psi_2 \\ \dot{L}_1 &= -x_1\psi_1(1-x_1) + x_1\psi_1 \left(\frac{\beta x_2}{x_1 + x_2} \right) + x_1\psi_1 u_1 + x_1\psi_1(1-2x_1) - x_1\psi_1 \beta \left(\frac{x_2}{x_1 + x_2} \right)^2 - x_1\psi_1 u_1 \dots \\ &\dots + x_2\psi_2 \gamma \left(\frac{x_1 x_2}{(x_1)^2} \right) \\ \dot{L}_1 &= L_1(1-x_1) - L_1 \left(\frac{\beta x_2}{x_1 + x_2} \right) - L_1(1-2x_1) + L_1 \beta \left(\frac{x_2}{x_1 + x_2} \right)^2 - L_2 \gamma \cdot \frac{x_2}{x_1} \\ \dot{L}_1 &= L_1(1-x_1) - L_1 \left(\frac{\beta x_2}{x_1 + x_2} \right) - L_1(1-x_1) + L_1 x_1 + L_1 \beta \left(\frac{x_2}{x_1 + x_2} \right)^2 - L_2 \gamma \cdot \frac{x_2}{x_1} \\ \dot{L}_1 &= L_1 x_1 - L_1 \beta \left(\frac{x_1 x_2 + (x_2)^2}{(x_1 + x_2)^2} \right) + L_1 \beta \left(\frac{(x_2)^2}{(x_1 + x_2)^2} \right) - L_2 \gamma \cdot \frac{x_2}{x_1} \\ \dot{L}_1 &= L_1 x_1 - L_1 \beta \left(\frac{x_1 x_2}{(x_1 + x_2)^2} \right) - L_2 \gamma \cdot \frac{x_2}{x_1} \\ \dot{L}_1 &= L_1 \left(x_1 - \beta \cdot \frac{x_1 x_2}{(x_1 + x_2)^2} \right) - L_2 \left(\frac{\gamma x_2}{x_1} \right)\end{aligned}$$

where $L_1(T) = -x_1(T) \cdot A < 0$.

Similarly, employing a change of variables using $L_2 = -x_2\psi_2$, then

$$\begin{aligned}
\dot{L}_2 &= -\dot{x}_2\psi_2 - x_2\dot{\psi}_2 \\
\dot{L}_2 &= -\left(x_2\left(\alpha - \gamma \cdot \frac{x_2}{x_1}\right) - u_2x_2\right)\psi_2 - x_2\left(\beta \cdot \left(\frac{x_1}{x_1+x_2}\right)^2\psi_1 - \left(\alpha - 2\gamma \cdot \frac{x_2}{x_1} - u_2\right)\psi_2\right) \\
\dot{L}_2 &= -x_2\psi_2\left(\alpha - \gamma \cdot \frac{x_2}{x_1}\right) + x_2\psi_2u_2 - x_1\psi_1\beta \cdot \frac{x_1x_2}{(x_1+x_2)^2} + x_2\psi_2\left(\alpha - 2\gamma \cdot \frac{x_2}{x_1}\right) - x_2\psi_2u_2 \\
\dot{L}_2 &= L_2\left(\alpha - \gamma \cdot \frac{x_2}{x_1}\right) + L_1\beta \cdot \frac{x_1x_2}{(x_1+x_2)^2} - L_2\left(\alpha - 2\gamma \cdot \frac{x_2}{x_1}\right) \\
\dot{L}_2 &= L_2\left(\alpha - \gamma \cdot \frac{x_2}{x_1}\right) + L_1\beta \cdot \frac{x_1x_2}{(x_1+x_2)^2} - L_2\left(\alpha - \gamma \cdot \frac{x_2}{x_1}\right) + L_2\gamma \cdot \frac{x_2}{x_1} \\
\dot{L}_2 &= L_1\left(\frac{\beta x_1x_2}{(x_1+x_2)^2}\right) + L_2\left(\frac{\gamma x_2}{x_1}\right)
\end{aligned}$$

where $L_2(T) = -x_2(T) \cdot B < 0$.

Therefore, the system of differential equations for the switching functions $L_1(t)$ and

$L_2(t)$ is

$$\begin{cases} \dot{L}_1 = L_1\left(x_1 - \beta \cdot \frac{x_1x_2}{(x_1+x_2)^2}\right) - L_2\left(\frac{\gamma x_2}{x_1}\right) \\ \dot{L}_2 = L_1\left(\frac{\beta x_1x_2}{(x_1+x_2)^2}\right) + L_2\left(\frac{\gamma x_2}{x_1}\right) \\ L_1(T) = -Ax_1(T), L_2(T) = -Bx_2(T) \end{cases} \quad (5)$$

It can be seen that $L_1(T) = -Ax_1(T) < 0, L_2(T) = -Bx_2(T) < 0$.

Now an important characteristic of the switching functions $L_1(t)$ and $L_2(t)$ is discussed

in Lemma 2.

Lemma 2. *Functions of switching $L_1(t)$ and $L_2(t)$ cannot be zero on any finite time interval $\Delta \in [0, T]$.*

Proof 2a. Assume the opposite is true, that is $L_1(t) \equiv 0$ for $t \in \Delta \in [0, T]$.

Then, $\dot{L}_1(t) \equiv 0, t \in \Delta$. From the first equation of system (5), it is obtained that

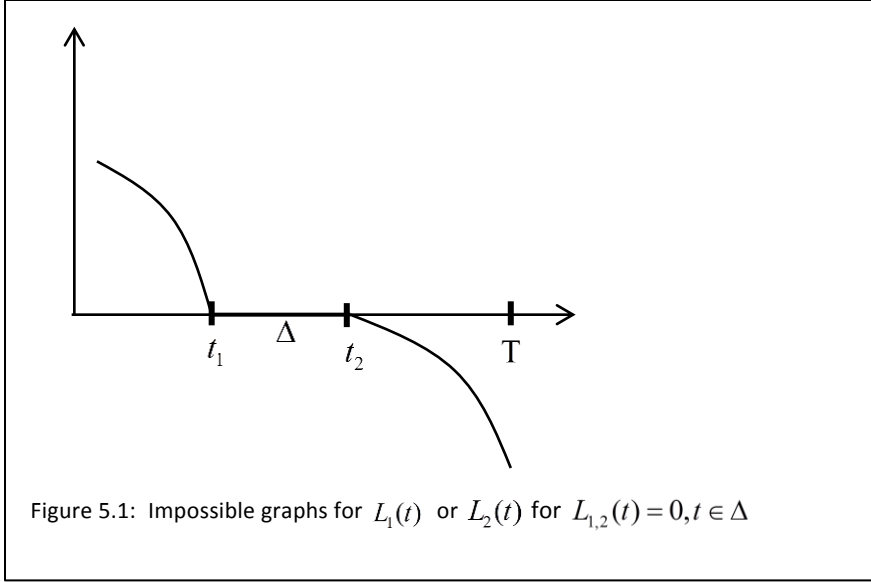
$L_2(t) = 0, t \in \Delta$. The second equation holds for interval Δ . Then $L_1(t) = 0$ and $L_2(t) = 0$ for the entire interval $[0, T]$ which contradicts the boundary condition and nontriviality of functions $L_1(t)$ and $L_2(t)$. Thus, the assumption is false, and the statement is proven.

Proof 2b. Assume the opposite is true, that is $L_2(t) \equiv 0$ for $t \in \Delta \in [0, T]$.

Then, $\dot{L}_2(t) \equiv 0, t \in \Delta$. From the second equation of system (5), it is obtained that

$L_1(t) = 0, t \in \Delta$. The first equation holds for interval Δ . Then $L_1(t) = 0$ and $L_2(t) = 0$ for the entire interval $[0, T]$ which contradicts the boundary condition and nontriviality of functions $L_1(t)$ and $L_2(t)$. Thus, the assumption is false, and the statement is proven.

Consequently, through Lemma 2, the singular control cannot be optimal and the optimal solutions do not contain singular arcs. Graphs for $L_1(t)$ or $L_2(t)$ such as shown in figure 5.1 would be impossible since switching functions $L_1(t)$ and $L_2(t)$ cannot be zero on any finite time interval.



Since for no finite interval $\Delta \in [0, T]$ in which $L_{1,2}(t) = 0$, from PMP, the optimal controls can be adjusted to

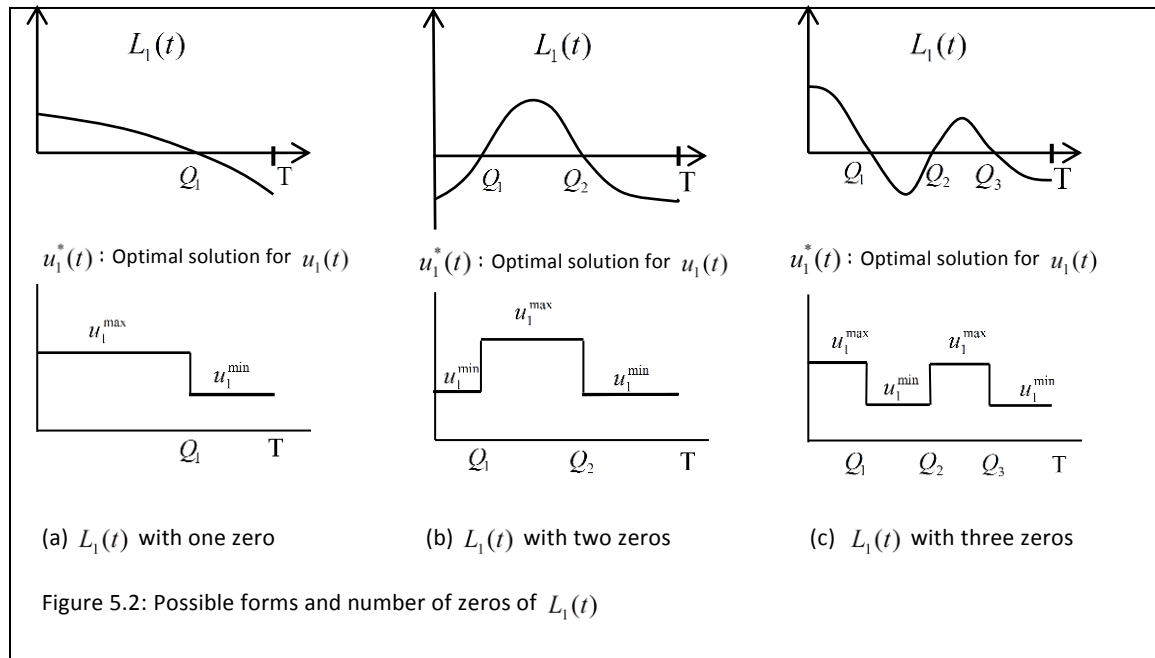
$$u_1^*(t) = \begin{cases} u_{\max}^1, & \text{if } L_1(t) > 0 \\ u_{\min}^1, & \text{if } L_1(t) < 0 \end{cases}$$

and

$$u_2^*(t) = \begin{cases} u_{\max}^2, & \text{if } L_2(t) > 0 \\ u_{\min}^2, & \text{if } L_2(t) < 0 \end{cases}$$

Moreover, $L_1(t)$ and $L_2(t)$ as was stated earlier are both negative at $t = T$ (the last moment) and functions $L_1(t)$ and $L_2(t)$ are continuous. Hence, the optimal controls must take the minimal values at least at the end of the time interval $[0, T]$. However,

until system (5) is solved using, for example, the methods developed in [18] – [22], the different number of zeros in the switching function can be recognized, and depending on the number of zeros, the different types of the optimal controls. In Figure 5.2, different possible types of the switching functions $L_{1,2}(t)$ and corresponding optimal controls are shown.



Further investigation with the analysis of the second and higher derivatives of the switching functions can be completed to relate to the number of zeros and draw more analytical conclusions. Upon successful completion, with at least the maximum possible number of switchings analytically found, then a very complex two point

boundary value problem for the PMP can be rewritten as one of the finite dimensional optimization. Papers of Grigorieva and Khailov show examples of this type of process [18] – [22].

For example, if it was analytically established that there were at most three switchings, then an optimization computer program can be written and the precise positions of the switchings in the optimal control would be found, and the maximum of the objective function would be evaluated. Since this model is quite complex (bilinear with two bounded control functions), the analytical solution of the problem was not the goal of the thesis.

In the next section, numerical methods will be attempted to solve system (1) for different types of the piecewise constant optimal control with at most two switching for selected objective functions, given by cases 1, 2 and 3. Results of the computer modeling of system (1), with one only control function $u_1(t)$, will be demonstrated and discussed.

CHAPTER VI

COMPUTER SIMULATIONS

With the degree of difficulty in analysis of the optimal control problem of the Holling Tanner model, a computer program in MAPLE that solves the problem numerically is advantageous. Some preliminary numerical simulations were done using a computer program in MAPLE to solve the optimal control problem for the predator-prey system (1) with only one control function $u_1(t)$. In order to simulate the predator-prey model with an assumption to harvest only the prey and with consideration that the optimal control has at most two switching on the time interval $[0, T]$ and ends in the minimal value (minimal harvesting rate), the program must contain $u_1(t)$ as a piecewise constant function of the type $u_{\min}, u_{\max}, u_{\min}$. The program of finite dimensional optimization calculates all possible values of the objective functions for all possible positions $0 \leq \tau_1 \leq \tau_2 \leq T$ over time interval $[0, T]$ and finds the maximum value of $J(u_1(t), u_2 = 0, \tau_1, \tau_2)$. Then, the program gives the exact value of the switching position, and therefore, the exact type of the optimal control.

Results of the computer modeling were obtained for three cases of the objective functions:

a) Case 1: $J(u_1(t), u_2 = 0) = x_1(T) \rightarrow \max_{u_1(\cdot) \in D(t)}$

Here, only population of the prey will be maximized at moment $t = T$ using the best harvesting strategy with at most two switching, ending at the minimal rate u_{\min}^1 .

b) Case 2: $J(u_1(t), u_2 = 0) = Ax_1(T) + Bx_2(T) \rightarrow \max_{u_1(\cdot) \in D(t)}$

In this instance, once more, only the prey will be harvested on the interval $[0, T]$ using the best harvesting strategy with at most two switching, ending at the minimal rate u_{\min}^1 , where $A = 1$ and $B = 1$, and the total number of both species will be maximized (saved) at terminal time T .

c) Case 3: $J(u_1(t), u_2 = 0) = Ax_1(T) + Bx_2(T) \rightarrow \max_{u_1(\cdot) \in D(t)}$

Yet again, only the prey will be harvested at moment $t = T$ using the best harvesting strategy with at most two switching, ending at the minimal rate u_{\min}^1 , but where $A = 0.7$ and $B = 0.2$, and the weighted total number of both species will be maximized (saved) at terminal time T .

6.1 Case 1. $J(u_1, u_2) = x_1(T) \rightarrow \max_{u_1(\cdot), u_2(\cdot) \in D(t)}$

Consider the case with only one control ($u_1(t)$ and $u_2(t) = 0$), using positive values for the parameters α, β, γ . This circumstance could be a signification situation if only the prey is going to be harvested. For instance, in the fishing industry perhaps only the food fish, i.e. crab, lobster, etc., are considered significant enough to harvest and the predator fish are not of substantial importance. With parameter values of $\alpha = 0.67, \beta = 1, \gamma = 1, u_{\min}^1 = 0.06$, and $u_{\max}^1 = 1$, graphs in figures 6.1-6.5 are presented that represent an optimal solution with the only one switching at the middle of the second day ($t = 2.5$). This shows that harvesting at the maximum rate should continue only for 2.5 days. At this point, it must be at the minimal rate u_{\min}^1 . It is interesting to note that the population does not reach zero so the prey would not die off. The graph in figure 6.1 shows how the value of the functional $J(u_1(t), \tau_1, \tau_2)$ is changing depending on the positions of two switchings $0 \leq \tau_1 = \tau_2$.

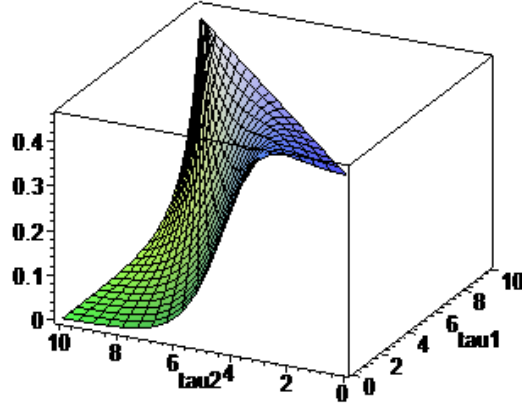


Figure 6.1 Case 1: Prey population at $t = T$ versus τ_1 and τ_2

It is interesting that though control $u_1(t)$ had originally two switchings, the maximum of the objective function $J(u_1(t), \tau_1, \tau_2)$ happened for the optimal harvesting rate with only one switching at time $t = 2.5$. For this situation, it means that if the prey population is maximized the prey population at the final moment $t = T$ by harvesting only prey species, the optimal strategy described above should be followed. This strategy would maximize the amount of the preys at the tenth day ($T = 10$) under the optimal harvesting strategy shown below, during the ten day time period.

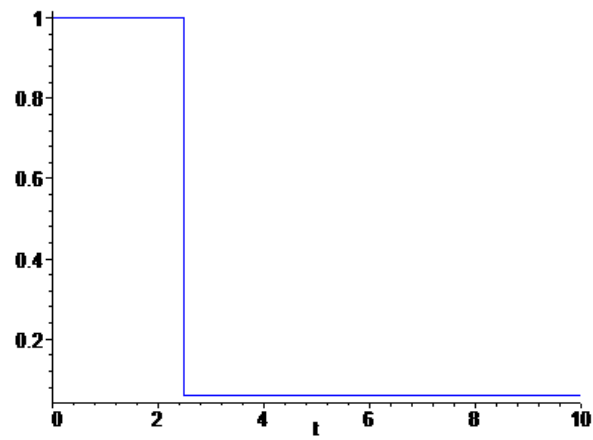


Figure 6.2 Case 1: Optimal control harvesting strategy

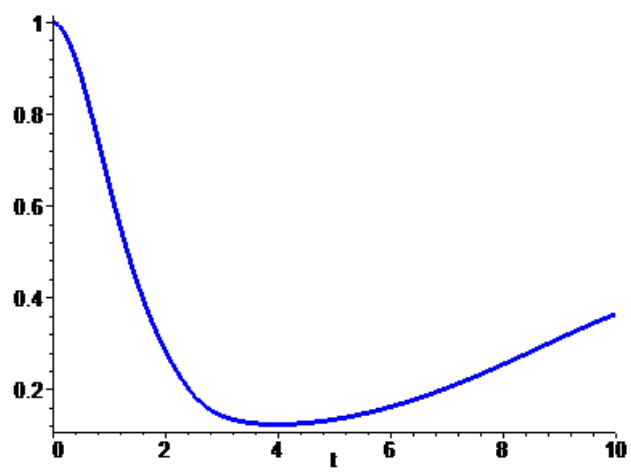


Figure 6.3 Case 1: Prey population over time

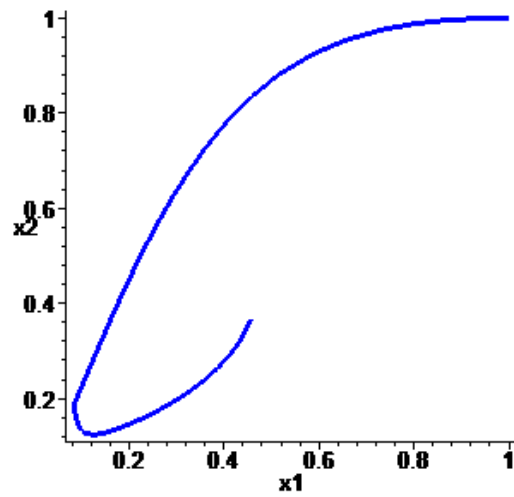


Figure 6.4 Case 1: Phase portrait of prey population

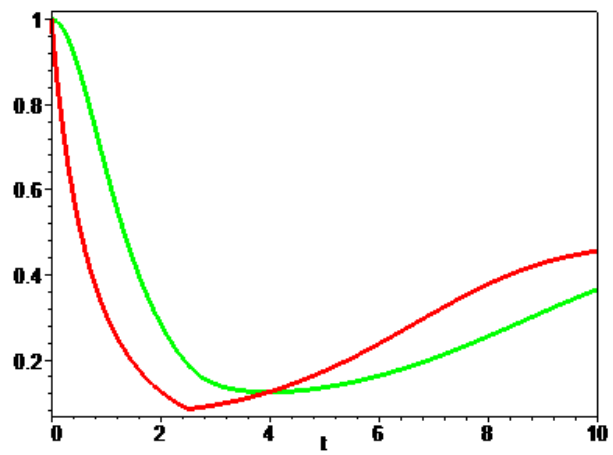


Figure 6.5 Case 1: Prey and Predator populations over time

6.2 Case 2. $J(u_1(t), u_2 = 0) = Ax_1(T) + Bx_2(T) \rightarrow \max_{u_1(\cdot) \in D(t)}$

Consider the case with again one control, $u_1(t)$, and $u_2(t) = 0$ (Predators are not caught during harvest), using positive values for the parameters α, β, γ , and where $A = 1$, $B = 1$, $u_{\min}^1 = 0.06$, and $u_{\max}^1 = 1$. The objective function indicates that the total number of the both species at terminal time T will be maximized (saved). Using the fishing industry as an example again, the harvest would consist of only the prey fish and not the predator fish. With parameter values of $\alpha = 0.67, \beta = 1, \gamma = 1$, graphs in figures 6.6-6.10 are presented that represent the optimal solution with the harvesting rate switching at the end of the first day ($T = 1$). This shows that harvesting at the maximum rate should continue only for one day. Then, fishing would be done for nine days at the minimal rate. Once again, it is interesting to note that the population does not reach zero so the prey would not die off. The graph in figure 6.6 shows how the value of the functional $J(u_1(t), \tau_1, \tau_2)$ is changing depending on the positions of two switchings $0 \leq \tau_1 = \tau_2$.

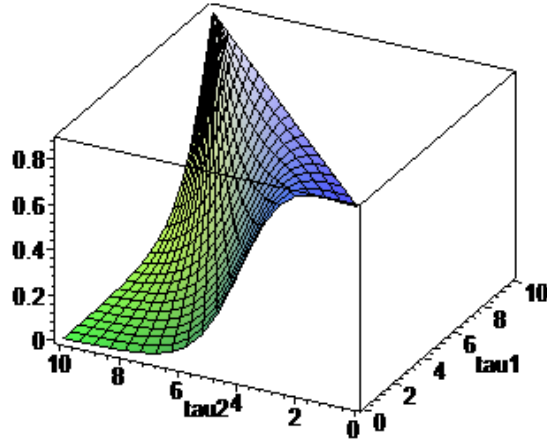


Figure 6.6 Case 2: Prey and predator population at $t = T = 10$ versus τ_1 and τ_2

Although control $u_1(t)$ had originally two switchings, the maximum of the objective function $J(u_1(t), \tau_1, \tau_2)$ happened for the optimal harvesting rate with only one switching at time $t = 1$. For this situation, it means that if the predator and prey populations are maximized at the final moment $t = T$ by harvesting only prey species, the optimal strategy described above should be followed. This strategy would maximize the amount of the predators and preys at the tenth day ($T = 10$) under the optimal harvesting strategy shown below, during the ten day time period.

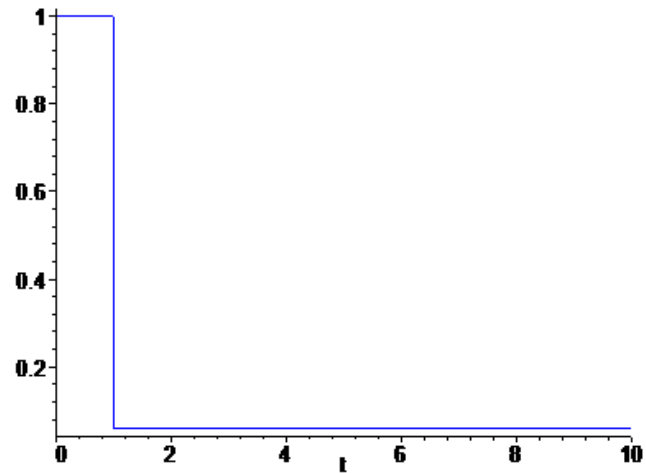


Figure 6.7 Case 2: Optimal control harvesting strategy

In this case, the minimal rate needs to be switched earlier than in case 1. The predator population is important since predators must eat prey, so if harvesting of prey continues for a longer period of time, there would be no food for the predators and $x_1(T) + x_2(T)$ would not be maximized.

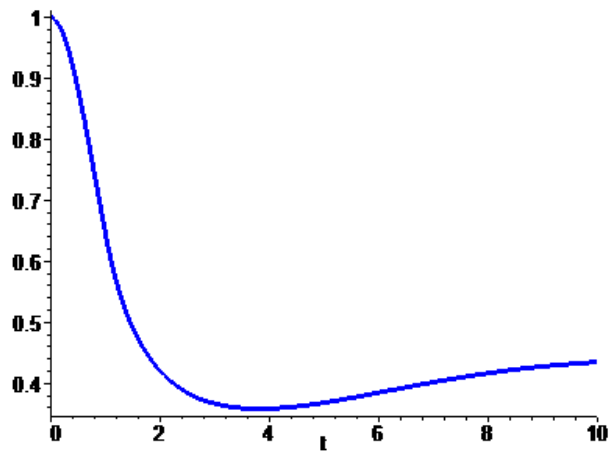


Figure 6.8 Case 2: Prey population over time

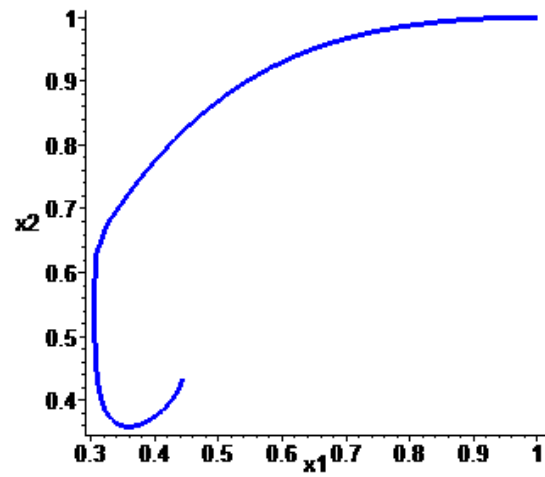


Figure 6.9 Case 2: Phase portrait of prey population

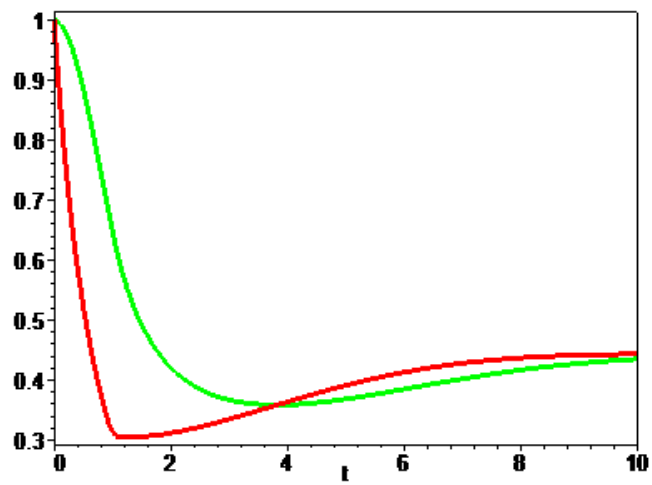


Figure 6.10 Case 2: Prey and Predator populations over time

6.3 Case 3. $J(u_1(t), u_2 = 0) = Ax_1(T) + Bx_2(T) \rightarrow \max_{u_1(\cdot) \in D(t)}$

Finally, consider the case with two controls, using positive values for the parameters

α, β, γ , and where $A = 0.7, B = 0.2, u_{\min}^1 = 0.06$, and $u_{\max}^1 = 1$. As stated earlier, A and

B are nonnegative constants, less than or equal to one, that can be interpreted as the

economic importance of certain species. Thus, for $B = 0.2$, the predator population is

much less important than the prey population where $A = 0.7$. With parameter values of

$\alpha = 0.67, \beta = 1, \gamma = 1$, graphs in figures 6.11-6.15 are presented that represent an

optimal solution over the period of ten days, which shows that harvesting at the

maximum rate should continue until the end of the second day. For a third time, it is

interesting to note that the populations do not reach zero so the prey and predators

would not die off.

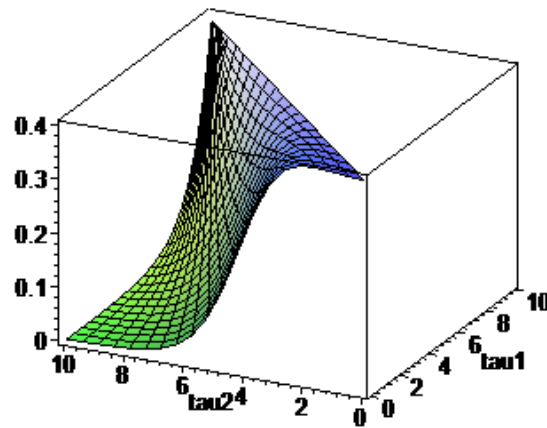


Figure 6.11 Case 3: Prey and predator population at $t = T = 10$ versus τ_1 and τ_2

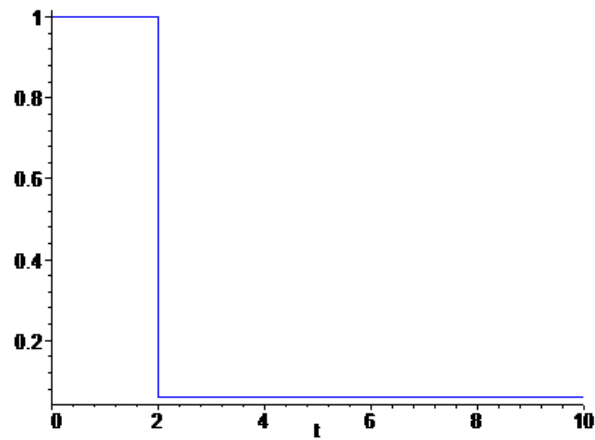


Figure 6.12 Case 3: Optimal control harvesting strategy

In this case, the minimal rate needs to be switched earlier than in case 1, but later than in case 2. The prey population is the more important value. However, the predator population is still of some importance again since predators must still eat prey to survive.

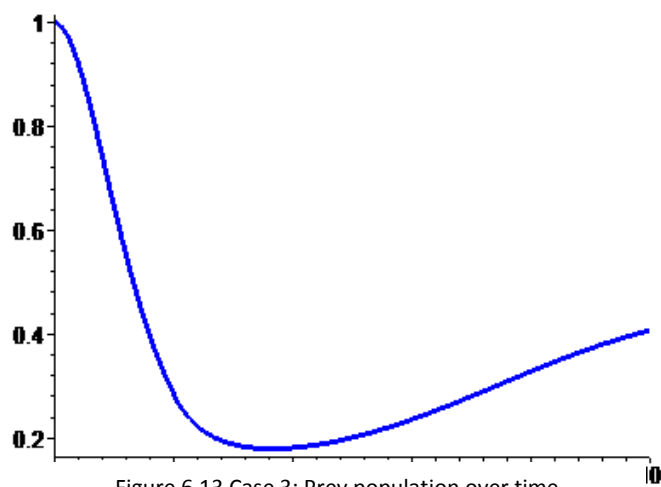


Figure 6.13 Case 3: Prey population over time

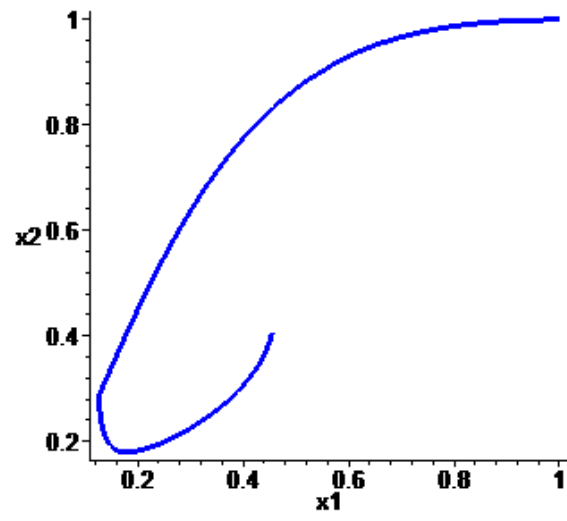


Figure 6.14 Case 3: Phase portrait

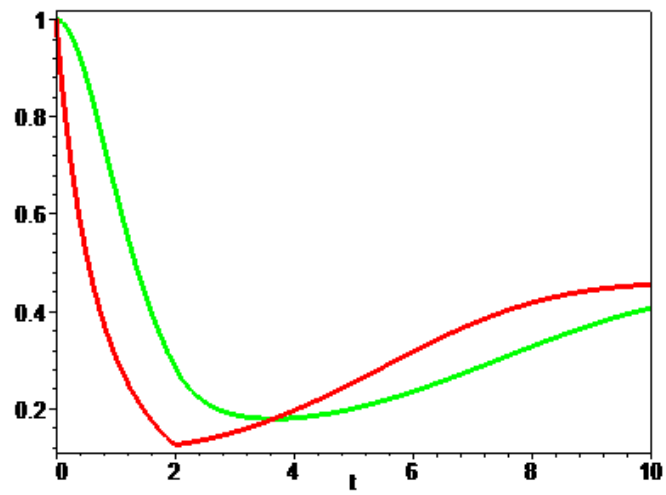


Figure 6.15 Case 3: Prey and Predator populations over time

CHAPTER VII

CONCLUSIONS

The following summarizes the work done:

1. Models used for predator-prey situations were reviewed, including the Malthusian model, the Verhulst-Pearl model, the Lotka-Volterra model, the Holling model, and the Holling-Tanner model.
2. The Verhulst-Pearl model, the Lotka-Volterra model, the Holling model, and the Holling-Tanner model were analyzed for the number and type of equilibrium points. Additionally, computer phase portraits were presented of the models.
3. An optimal control problem was formulated for the Holling-Tanner model and was investigated to determine important characteristics of the optimal controls and optimal trajectories using the methods of the switching functions.
4. An optimization computer program in MAPLE was employed to simulate our model numerically for certain type of the possible control and in order to determine the optimal solution that maximizes three types of the objective functions. Results of different outputs are discussed.

The work done in this thesis lends itself to further investigation and expansion. Continuing work on the Holling-Tanner model could be done by investigating the second and higher derivatives of the switching functions of system (5) by finding the maximum number of the switchings. This way, a very complex two point boundary value problem for the maximum principle (systems (1) and (5)), would be reduced to one of the finite dimensional optimization. Then, the best optimal strategy (optimal control) with precise positions of the optimal control for the model would be found numerically. One can assume that this would help the fishing industry to plan their harvesting policy over given time interval with minimal harm to the ecological systems. As the need for help with ecosystems continues to grow, mathematical studies with optimal control in this area will be of importance.

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