A THEORETICAL RELATIONSHIP BETWEEN MATHEMATICS AND MECHANICS

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To the Dean of the Graduate School:

I am submitting herewith a thesis written by Kelli Wellborn entitled "A Theoretical Relationship Between Mathematics and Mechanics." I have exampled this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Masters of Science with a major in Mathematics.

Don E. Edwards

Dr. Don Edwards, Major Professor

We have read this thesis and recommend its acceptance:

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ABSTRACT

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A THEORETICAL RELATIONSHIP BETWEEN MATHEMATICS AND MECHANICS

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Classical mathematics began during the Egyptian and Babylonian time and solved problems analytically with no proof of theories being performed. In the late eighteenth century, theoretical problems began emerging into traditional mathematics where the theoretical approach to classical problems began to be explored. Traditional mathematics included group, field and ring theory and can be applied to other subjects such as molecular symmetry in chemistry. Traditional mathematics expanded and provided discovery of the newest field of study, idempotent mathematics. Idempotent mathematics emerged in the nineteenth century stemming from the definition of an idempotent element and an algebraic structure known as a semiring. Idempotent and traditional mathematics have been said to have a correspondence to each other just like quantum mechanics and classical mechanics do through Neil Bohr's correspondence principle. Using Erwin Schrödinger's particle in a box experiment, the beginning steps were taken to find the theoretical relationship between the two subjects.

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CHAPTER I

INTRODUCTION

Historians can date the beginning of mathematics back to around 3000 BC with the Egyptians and Babylonians, referred to as the era of classical mathematics. Classical mathematicians focused on finding the roots of polynomial equations and how to notate the equations and roots as well as other subjects such as geometry and number theory (Kleiner, 2007). Both the Egyptians and Babylonians used types of images to symbolize a number in numeric problems. The Egyptians used symbols known as hieroglyphics, shown in Figure 1 below, while the Babylonians used wedges. The Babylonian number system contained two types of wedges, a vertical wedge, 𒁹, representing ones and a corner wedge, 𒌋, representing tens (Lewinter & Widulski, 2002; O'Connor & Robertson).

1	𓏺	10,000	𓂭
10	𓎆	100,000	𓆐
100	𓍢	1,000,000	𓁨
1,000	𓆼		

Figure 1: Egyptian Hieroglyphics courtesy of O'Connor & Robertson

After the Babylonian and Egyptian era, Greek mathematicians began to emerge using the astronomical and mathematical ideas from the Egyptians and Babylonians. Greek mathematics focused mainly on geometry and founded many of the geometric properties and proofs taught today. Greek mathematicians used geometric means to solve proofs by starting with known properties of geometry and using deductive reasoning to end with what they were trying to prove (Lewinter & Widulski, 2002; "Mathematical proof," n.d.). The Greek mathematicians had strong skills toward geometry, while the Islamic mathematicians focused on algebra.

Islamic mathematicians possessed very strong skills in algebra and due to these skills, the creation of "a systematic study of methods for solving quadratic equations" emerged ("Algebra," 2013). The term algebra comes from an Islamic book written by the Euclid of algebra, Al-Khwarizmi (Kleiner, 2007, p.3). This book included many solutions to quadratic and cubic polynomial equations as well as some geometric proofs alongside the solutions. The need to create a general definition came about from the geometric proofs handed down and contributed to the transition of classical mathematics to a new era known as traditional mathematics.

Traditional mathematics includes theories such as group, field and ring theory referred to as abstract algebra. In the eighteenth century, group theory emerged, which is the study of groups. A group is defined to be a set with a binary operation that is associative and contains an identity element and each element has an inverse (Nicholson, 2007).

Groups later evolved due to the addition of properties and restrictions such as those present in fields and rings. Field and ring theory coexisted together around the same time period.

A field is defined to be a set of elements that contains two binary operations, addition and multiplication, is a commutative group under multiplication and addition and multiplication is distributive over addition. A ring is defined to be an additive commutative group, and multiplication is distributive over addition, commutative, associative and contains an identity element (Clark, 1984). Due to the evolution of traditional mathematics, group theory can be applied to subjects such as molecular symmetry of compounds.

Molecular symmetry contains five symmetry elements: identity (E), n-fold axis of symmetry (C_n), reflection plane (σ), the inversion center (i) and the improper rotation axis (S_n). Molecular symmetry allows for the determination of chirality, which is the capability to superimpose on its mirror image or not. Based on the knowledge of the appearance of an improper rotation, one can say that a molecule is achiral (Atkins & de Paula, 2006). Traditional mathematics contains several subcategories such as semirings, which led to the discovery of a newer field in mathematics known as idempotent mathematics.

Idempotent mathematics emerged due to the work of two Russian mathematicians, Viktor Pavlovich Maslov and Grigory Litvinov. Idempotent refers to a property whereby an element of a set is unchanged in value when multiplied or otherwise operated on by itself, shown as a*a = a ("Idempotent", 2012). An area in idempotent mathematics that is studied incorporates an algebraic structure referred to as a semiring. Semirings contain the property of associativity for addition and multiplication, commutative property of addition, an identity element, distributive property, and an absorbing element (Glazek, 2002).

A semiring is known as an idempotent semiring if any element that exists in the semiring is unchanged in value when operated on itself under tropical addition, defined by $a \oplus a = a$ (Ellis, 2005). Some of the best-known idempotent semirings include the max-plus and min-plus semirings.

Max-plus semiring is denoted as $\mathbb{R} \cup \{-\infty\}$ with operations $\oplus$ and $\otimes$ defined by $a\oplus b: = \max(a,b)$ and $a\otimes b: = a+b$. The first notation, $a\oplus b: = \max(a,b)$, can be translated into the tropical addition of a and b is equivalent to the maximum value a or b in algebra (Hebisch & Hanns, 1998). The second notation, $a\otimes b: = a+b$, can be translated into the tropical multiplication of a and b is equivalent to the addition of a and b in algebra. Another type of idempotent semiring is known as a min-plus semiring. The min-plus semiring is $\mathbb{R} \cup \{\infty\}$ equipped with $a\oplus b: = \min(a,b)$ and $a\otimes b: = a+b$ (Golan, 2005). Min-plus can be applied to a type of problem known as the shortest path problem, which determines the quickest amount of time or shortest amount of distance it takes to get from one point to another through multiple points. The shortest path problem uses matrix multiplication and addition under the conditions of the min-plus semiring to determine the shortest path.

The discovery of idempotent mathematics came about when Maslov studied idempotent semirings and used semirings to transform a non-linear function in differential equations to a linear function under idempotent semirings (Litvinov, 2007). One example from Maslov and Litvinov involved the derivation of a special case of the Hamilton-Jacobi equation from the heat equation under the properties of an idempotent semiring. The transformation of a non-linear function to a linear function is referred to as Maslov's dequantization also known as the idempotent correspondence principle (Litvinov, 2007). Using the idempotent correspondence principle, Maslov and Litvinov showed an idea where the relationship between quantum mechanics and classical mechanics is analogous to the relationship between traditional mathematics and idempotent mathematics.

Due to the evolution of classical mathematics to idempotent mathematics, development in different number systems and proofs supports the explanation of why algebraic laws hold true. Idempotent mathematics is a growing field of study and can be applied to theoretical computer science and mechanics through analogous principles. In the near future, the hope for a connection between mechanics and mathematics will be more prominent using experiments such as the Schrödinger equation and the Markov chain under the conditions of the max-plus and min-plus semirings.

CHAPTER II

A BRIEF HISTORY

Traditional mathematics also referred to as modern mathematics, surfaced in the nineteenth century as a transition from classical mathematics. During the classical period, mathematicians studied number theory, geometry, properties of various number systems, and the solution of polynomial equations (Kleiner, 2007). The Egyptians (3000 BC to 300 BC) became some of the first known mathematicians and represented numbers with symbols known as hieroglyphics.

The Egyptians created an interesting way to multiply numbers using the dyadic method, meaning doubling. The Egyptians would start with two columns. The left column would begin with the integer one, and that number would be doubled until any combination of the doubled numbers in the left column could be summed together to the smaller of the two numbers being multiplied. The right column begins with the larger number being multiplied. This number would be doubled the same amount of iterations as the left column. To get the product, pick the numbers in the right column that correspond to the numbers in the left column that added up to the smaller number being multiplied. The sum of the corresponding numbers in the right column represents the product of the two numbers; an illustrated example follows below in figure 2 (Lewinter & Widulski, 2002).

20 * 32

1	32
2	64
4*	128*
8	256
16*	512*

16 + 4 = 20

20 * 32 = 128 + 512 = 640

Figure 2: Egyptian's dyadic method example

The Babylonians (1800 BC to 1600 BC) expressed numbers with vertical and horizontal wedges. The Babylonians did not use the decimal system being used today; instead they used a sexagesimal scale, which has a base 60 instead of base 10 (Lewinter & Widulski, 2002). Borrowed from the Babylonians, this scale would be later introduced into Greek astronomical and mathematical calculations in the sixteenth century.

Using ideas from the Egyptians and Babylonians, the Greeks applied themselves in the subjects of astronomy and geometry. Greek mathematicians demonstrated strong skills in geometric algebra. Thales, a Greek philosopher, emphasized the importance of proving mathematical truths and used geometry to do so. Diophantus of Alexandria, a Greek mathematician, introduced some solutions of equations in integers or rational numbers along with partial algebraic notation (Kleiner, 2007). Partial algebraic notation denoted unknowns, negations, equalities, squares, cubes and other mathematical objects as symbols (Kleiner, 2007). One of the most famous geometric contributions included Euclid's work *Elements*, which contained "geometric propositions that, if translated into algebraic language, yield algebraic results: laws of algebra as well as solutions to quadratic equations" (Kleiner, 2007, p.2).

Shifting from a geometric focus, Islamic mathematicians focused mainly on algebraic problems and equations. Al-Khwarizmi studied algebraic equations and how to solve for the variables through canceling out numbers and variables from each side of the equation. Thanks to the contributions of Al-Khwarizmi's book, *al-jabr w al-muqabalah*, mathematicians had different types of solutions to quadratic equations as well as geometric justifications (Kleiner, 2007). Due to these well-known mathematical contributions, the transition from classical mathematics to traditional mathematics had begun to emerge.

In the sixteenth and seventeenth century, two mathematicians, François Viète and René Descartes, introduced the use of symbols in mathematical notation. Viète distinguished the difference between arbitrary parameters, represented by consonants (b,c,d,f…), and variables, represented by vowels (a, e, i, o, u). Although Viète introduced symbols in equations, the equations contained partly symbolic characters and required that the algebraic expressions be written in the same degree; this created complications (Kleiner, 2007). Even with certain drawbacks, Viète's contributions accompanied the development in analytical geometry and calculus. The complications of Viète 's work became the center of study for French mathematician Descartes.

Using fully symbolic notation and explaining the basic elements of analytic geometry, Descartes improved Viète's work. Instead of using consonants and vowels, Descartes used letters at the beginning of the alphabet as parameters (a, b,c,…) and letters at the end of the alphabet as variables (x, y, z,…).

These two mathematicians "shifted the focus of attention from the solvability of numerical equations to theoretical studies of equations with literal coefficients" (Kleiner, 2007, p.10).

After Descartes, polynomial equations continued being studied more in depth along with the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra states, "every polynomial equation having complex coefficients and degree (greater than or equal to one) has at least one complex root" (Weisstein, Fundamental Theorem of Algebra). The solutions to polynomial equations aided in the transition to the study of properties of number systems.

In 1830, an English mathematician, George Peacock, published *Treatise of Algebra*; this book distinguished between arithmetical algebra and symbolic algebra. Arithmetical algebra refers to operations of addition, subtraction, multiplication and division on symbols that stood only for positive numbers; symbolic algebra refers to operations with symbols that obey the laws of arithmetical algebra and does not reference a specific number or object (Peacock, 1830). Shortly after Peacock, traditional mathematics started to emerge more in depth and extended into number theory and abstract algebra.

Traditional mathematics began to arise starting with group theory and extended further to field and ring theory. Traditional mathematics consists of subcategories within each theory such as subgroups, semigroups, subfields and semirings.

The transition to traditional mathematics came into existence largely due to mathematicians not being able to solve classical problems, thus began the study of abstract and axiomatic systems (Kleiner, 2007).

Around 1830, a French mathematician Évariste Galois, used the word "group" for the first time and became known as one of the founders of group theory. Several mathematicians started the concept of groups before Galois, however, the definitions were not used in a general sense. Continuing research in groups, Galois unintentionally began to uncover field theory while solving polynomial equations. Shadowing Galois's ideas, Richard Dedekind and Leopold Kronecker both studied and introduced different definitions for a field and continued to study beyond groups and fields (Kleiner, 2007). Dedekind began to research ring theory and introduced some fundamental properties of rings later to be used for the first abstract definition of a ring. Soon after the introduction of rings, Dedekind started to research a specific type of ring, known as a semiring.

Dedekind introduced a non-trivial example of semirings in 1894 in connection with the algebra of ideals of a commutative ring; semirings became an independent study for many algebraists such as Harry Schultz Vandiver in 1934 (Golan, 2005). Vandiver introduced the notation of a semiring and its acceptance as a fundamental algebraic structure, and in 1960, Samuel Eilenberg used semirings to research the theory of automata language in computer science. In 1993, Alexander Barvinok used the theory of semirings to apply to the theory of optimization and convex polytopes in the space of R^n_{min} (Golan, 2005).

Semirings began to be applied to a new field of mathematics called idempotent mathematics, recently discovered due to mathematicians Viktor Pavlovich Maslov and Grigory Litvinov (Golan, 2005; Litvinov & Maslov, 2003).

Idempotent mathematics became a field of traditional mathematics that bases its foundation on using idempotent semirings and semifields to solve applied problems in computer science and discrete mathematics. Idempotent mathematics derived from the definition of an idempotent element in abstract algebra. An idempotent element can be defined as "a mathematical quantity which when applied to itself under a given binary operation equals itself" (Idempotent, 2012). Idempotent mathematics replaces simple algebraic operations such as multiplication and addition with basic operations such as minimum and maximum.

From the Egyptians to current mathematicians, mathematics has evolved from arithmetic to theoretical algebra and currently expands into other subjects such as mechanics, computational chemistry and computational biology. The history of mathematics looks that of a continuous sinusoidal wave with oscillating eras and infinite research and discovery.

CHAPTER III

EVOLUTION OF TRADITIONAL MATHEMATICS

Group Theory

Due to French mathematician Joseph Louis Lagrange, the evolution of traditional mathematics started with the introduction of group theory in 1770. Lagrange expanded on the known methods of Viète and Descartes for solving cubic and quadratic equations. Lagrange attempted to do an analysis on equations with degree n to polynomial equations. Although Lagrange's work did not resolve the problem at the time, this became the first time that the solutions of polynomial equations and the permutations of groups coexisted with each other (Kleiner 2007; Kleiner 1986). Even though Lagrange did not coin the term "group" or define a group, the work on polynomial equations started the "group" concept.

Johann Carl Friedrich Gauss became the next mathematician to use the concept of a group in 1801. Explicit examples of groups can be seen in geometry and analysis; however, Galois would come along in latter years to be known as the founder of group theory (Kleiner, 2007).

Mathematicians observed Galois' work and one notable mathematician, Arthur Cayley, provided the first abstract definition of a group:

"A set of symbols, 1, α, β, ..., all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a group... These symbols are not in general convertible [commutative], but are associative and it follows that the entire group is multiplied by any one of the symbols, either as further or nearer factor..., the effect is simply to reproduce the group" (Kleiner, 1986, p.208).

The definition of a group has evolved throughout the years and currently exists as the definition shown below:

A group is a nonempty set G with a binary operation that satisfies the following properties:

1. G is closed under the binary operation;
2. The operation is associative;
3. There is an identity element in G;
4. Every element of G has an inverse in G.

(Nicholson, 2007, p.73).

An example of a group would be the set of integers denoted by $\mathbb{Z}$, which represents discrete numbers from negative infinity to positive infinity. Under the binary operation of addition, every integer operated on another integer yields an element that exists in $\mathbb{Z}$. The element 0 serves as an identity element. For an element a, $-a$ is the inverse. In following years, the development of other properties helped with the exploration of field and ring theory.

Ring Theory

Ring theory developed from two broad categories, non-commutative and commutative rings, which developed from different sources in mathematics. Israel Kleiner describes the formation of commutative rings and non-commutative rings in figure 3 below (2007, p.42).

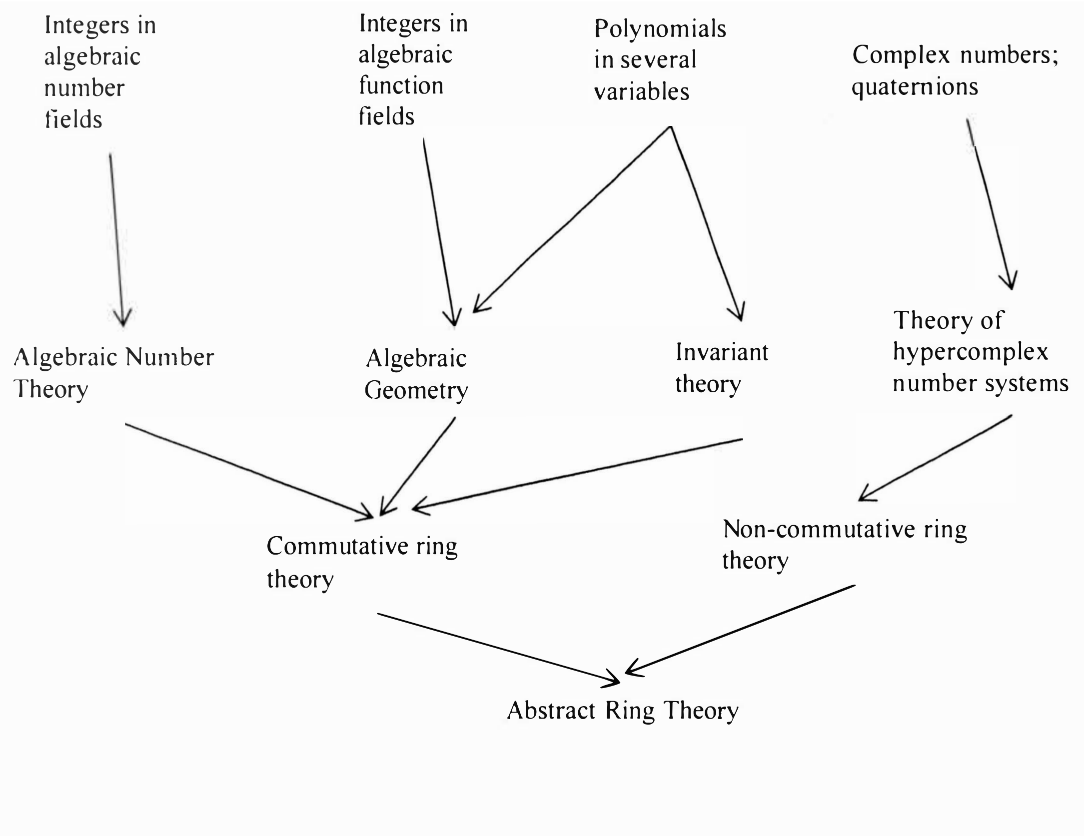


Figure 3: The development of Ring Theory

William Rowan Hamilton introduced a non-commutative ring from a single example of quaternions in 1843. Quaternion numbers take the form of $w + xi + yj + zk$, where w, x, y *and* z represent real numbers and i, j *and* k represent imaginary units and satisfy the condition $i^2 = j^2 = k^2 = ijk = -1$ (Weisstein, Quaternion). These numbers obey all algebraic laws of associativity and distributivity excluding the commutative property of multiplication, shown below.

$$\begin{array}{ll}
ijk = -1 & jk = i \\
i(ijk) = i(-1) & (ik)(ij) = i \\
-1(jk) = -i & -1(kj) = i \\
jk = i & kj = -i
\end{array}$$

$$\begin{array}{ll}
ijk = -1 & ik = j \\
j(ijk) = j(-1) & (jk)(ij) = j \\
-1(ik) = -j & -1(ki) = j \\
ik = j & ki = -j
\end{array}$$

$$\begin{array}{ll}
ijk = -1 & ij = k \\
k(ijk) = k(-1) & (jk)(ik) = k \\
-1(ij) = -k & -1(ji) = k \\
ij = k & ji = -k
\end{array}$$

Quaternions belong to a number system known as the hypercomplex number system. One example of hypercomplex number systems at work can be seen in a subject known as matrix algebra. Matrix algebra involves the addition and multiplication of matrices. Matrix addition follows all algebraic laws, however, matrix multiplication follows all of the algebraic laws except for the commutative property of multiplication (Bronson & Costa, 2009).

The hypercomplex number system contributed to the introduction of a definition for a non-commutative ring, to later be applied to the abstract definition of a ring.

Looking at figure 3 on page 14, the definition of a commutative ring began with integers in algebraic number fields, integers in algebraic function fields and polynomials in several variables. Integers in algebraic number fields extended into algebraic number theory, and later contributed to the definition of a commutative ring.

Integers in algebraic function fields and polynomials in several variables extended to the study of algebraic curves known as algebraic geometry. In algebraic geometry, algebra symbolizes the ring of polynomials and the geometry symbolizes the set of zeros of polynomials called an algebraic variety (Rowland, Algebraic Geometry). Polynomials of several variables contributed to both algebraic geometry as well as invariant theory. Invariant theory deals with the explicit definition of polynomial functions that do not change under transformations (Invariant, n.d). These contributions above, laid the framework for the definition of commutative rings.

When the first abstract definition for a ring came about, the existence of an abstract definition for a group had existed for about two decades and the definition of a field started to emerge. Israeli mathematician, Abraham Fraenkel, introduced the first abstract definition using the ideas of Kurt Hensel (Kleiner, 2007). Fraenkel defined a ring as a system R "on which two abstract operations are postulated: addition and multiplication. The first operation is assumed to satisfy the axioms of a group, and the second one is assumed to be associative and distributive with respect to the addition.

Further, R is assumed to contain at least one identity element relative to the second operation" (Corry, 2000, p.5-27). Fraenkel's definition contains all properties that the modern definition does with the exception of two axioms, which do not appear in the modern definition.

After several years of research in ring theory, the modern abstract definition of a ring follows:

> A ring is a set R with two binary operations, addition (+) and multiplication (*) that has the following properties:
>
> 1. Addition is associative and commutative.
> 2. There exist additive and multiplicative identity elements in R.
> 3. There exists an additive inverse for each element.
> 4. Multiplication is associative
> 5. Multiplication is distributive over addition
>
> $a * (b + c) = a*b + a*c$
>
> $(a + b)*c = a*c + b*c$
>
> (Nicholson, 2007, p. 157)

The definition of a ring came about due to the placement of restrictions such as the commutative property of addition and the property of distributivity on the definition of a group. Using the example from group theory, the set of integers satisfies the requirements for a ring.

Rings can be placed in subcategories such as subrings, polynomial rings, quotient rings and semirings. The expansion of semirings will be described in chapter four along with its contributions to a new field of study, idempotent mathematics.

Field Theory

In the nineteenth century, two mathematicians, Galois and Niels Henrik Abel, introduced field theory, which began to surface shortly after the founding of group theory. During the development of the abstract definition of a field, Dedekind introduced a more conceptually oriented definition of a field being an "infinite system of real or complex numbers so closed in itself and perfect that addition, subtraction, multiplication, and division of any two of these numbers again yields a number of the system" (Kleiner, 2007, p. 66). After Dedekind introduced the above definition, Leopold Kronecker presented a more algorithmic definition than Dedekind's. Kronecker defines a field as "the domain of rationality (R', R'', R''',...) contains...every one of those quantities which are rational functions of the quantities R', R'', R''',... with integer coefficents" (Kleiner, 2007, p.67). After Dedekind and Kronecker, a German mathematician, Heinrich Martin Weber introduced the first abstract definition of a field that consists of two types of composition, the first of which may be called addition, the second multiplication with the following restrictions:

1. *We assume that both types of composition are commutative.*
2. *Addition shall generally satisfy the conditions, which define a group.*
3. *Multiplication is such that*

   $a(-b) = -(ab)$

   $a(b + c) = ab + ac$

   $ab = ac$ imples $b = c$, unless $a = 0$

   Given b and c, $ab = c$ determines a, unless $b = 0$.

   (Kleiner, 2007, p. 72)

After centuries of research and development, below shows the modern definition used:

> A field is a set F with two binary operations, addition (+) and multiplication (*), such that if a, b and c are elements of F then the following properties follow:
>
> 1. Satisfies all conditions of a group under both operations.
> 2. Contains the commutative property of addition and multiplication.
> 3. Multiplication is distributive over addition
>
> $a^* (b + c) = a^*b + a^*c$
>
> $(a + b)^*c = a^*c + b^*c$
>
>   (Clark, 1984; Nicodemi, 2007)

Taking the example used for group theory, the set of integers does not satisfy the properties of a field because the inverse element does not exist under multiplication. However, the set of real numbers, denoted by $\mathbb{R}$, satisfies the conditions for a field.

Field and ring theory began to emerge due to the addition of properties on the definition of a group. Groups have been applied to many subjects such as mechanics and molecular symmetry used in chemistry. An application of traditional mathematics can be shown in chapter 5 using molecular symmetry as an example.

CHAPTER IV

IDEMPOTENT MATHEMATICS

To understand the field of idempotent mathematics, the origins of idempotent mathematics and semiring needs to be explored. The word idempotent originated in 1870 due to American mathematician Benjamin Peirce who first introduced the term idempotent in his work *Linear Associative Algebra*. Idempotent elements can be seen in a mathematical subject referred to as matrix mathematics. Idempotence can be defined as an element of a set that is unchanged in value when multiplied or otherwise operated on by itself shown as a*a = a (Idempotent, 2012). An example of this operation in matrix mathematics is as follows.

$$\textit{Let matrix } \boldsymbol{A} \textit{ be } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\textit{Show that } \boldsymbol{A} * \boldsymbol{A} = \boldsymbol{A}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1*1+1*0 & 1*1+1*0 \\ 0*1+0*0 & 0*1+0*0 \end{bmatrix} = \begin{bmatrix} 1+0 & 1+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The idempotent concept can be used in a field of study in traditional mathematics known as semirings.

A semiring is a set S with two binary operations, addition (+) and multiplication (*), such that if a, b and c are elements of S then the following properties follow:

1. Addition is associative and commutative
2. Multiplication is associative
3. There exists an identity element in S with respect to addition and multiplication
4. Multiplication is distributive over addition
5. There exists an absorbing element 0 such that

   a*0 = 0 = 0*a

   (Litvinov & Masolv, 2003; Golan, 2005).

A semiring is called commutative if it possesses the commutative property of multiplication, a*b = b*a. A semiring is called an idempotent semiring if it possesses the idempotent property, a $\oplus$ a = a for all a that exists in the semiring (Litvinov & Maslov, 2003). Some simple examples of semirings include the set of natural numbers, integers, and real numbers.

The best-known example of an idempotent semiring is the max-plus semiring and min-plus semiring also known as tropical algebra. The max-plus and min-plus semirings are defined on page four in chapter one. A max-plus semiring must satisfy all of the properties of a semiring explained above, however, a couple of the properties need more clarification.

The identity element under tropical addition for $\mathbb{R} \cup \{-\infty\}$ is $-\infty$ since $-\infty \oplus a = a = a \oplus -\infty$ for all a. The identity property is denoted as $0 \oplus a = a = a \oplus 0$. (Livinov &

Maslov, 2003). The identity element under tropical multiplication for $\mathbb{R} \cup \{-\infty\}$ is 0 since $0 \otimes a = a = a \otimes 0$ for all a. The identity property is denoted as $\mathbb{1} \otimes a = a = a \otimes \mathbb{1}$ (Litvinov& Maslov, 2003, p.3). The properties now become the following when describing the properties for a max-plus semiring:

1. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
2. $\mathbb{0} \oplus a = a = a \oplus \mathbb{0}$
3. $a \oplus b = b \oplus a$
4. $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
5. $\mathbb{1} \otimes a = a = a \otimes \mathbb{1}$
6. $\mathbb{0} \otimes a = \mathbb{0} = a \otimes \mathbb{0}$
7. $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
8. $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$

(Farlow, 2009)

In the example used for min-plus semiring in chapter five, tropical multiplication and addition of square matrices can be used to determine the shortest amount of time or distance it takes to get from one point to another. Tropical addition of matrices is denoted as $\mathbf{A} = \mathbf{X} \oplus \mathbf{Y}$ where $\mathbf{A} = [a_{ij}]$, $\mathbf{X} = [x_{ij}]$ and $\mathbf{Y} = [y_{ij}]$ where i and j goes from one to m. When performing tropical addition, the minimum of the elements in $\mathbf{X}$ and $\mathbf{Y}$ become the new element in $\mathbf{A}$. Looking at a four by four matrix, $\mathbf{A} = \mathbf{X} \oplus \mathbf{Y}$ would look like the following:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \oplus \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{bmatrix}$$

$$= \begin{bmatrix} \min(x_{11}, y_{11}) & \min(x_{12}, y_{12}) & \min(x_{13}, y_{13}) & \min(x_{14}, y_{14}) \\ \min(x_{21}, y_{21}) & \min(x_{22}, y_{22}) & \min(x_{23}, y_{23}) & \min(x_{24}, y_{24}) \\ \min(x_{31}, y_{31}) & \min(x_{32}, y_{32}) & \min(x_{33}, y_{33}) & \min(x_{34}, y_{34}) \\ \min(x_{41}, y_{41}) & \min(x_{42}, y_{42}) & \min(x_{43}, y_{43}) & \min(x_{44}, y_{44}) \end{bmatrix}$$

The notation for the tropical addition of matrices can be written as $a_{ij} = \min(x_{ij}, y_{ij})$ where *i* and *j* go from one to *m* for each element of **A**. Tropical multiplication of matrices is denoted as **A** = **X** ⊗ **Y** where **A** = $[a_{ij}]$, **X** = $[x_{ij}]$ and **Y** = $[y_{ij}]$ where *i* and *j* goes from one to *m*. When performing tropical matrix multiplication, tropical addition is performed at the same time. A simple matrix multiplication will be observed first and tropical multiplication will be applied to see how different the two operations can be.

The definition of matrix multiplication follows for an element of **A** (Bronson & Costa, 2009):

$$a_{ij} = (x_{i1} \times y_{1j}) + (x_{i2} \times y_{2j}) + \cdots + (x_{in} \times y_{nj}) = \sum_{n=1} x_{in} y_{nj}$$

In tropical matrix multiplication, instead of the summation, it becomes the minimum value of the sums of elements in **X** and **Y** and is denoted as (Farlow, 2009):

$$(x_{i1} \otimes y_{1j}) \oplus (x_{i2} \otimes y_{2j}) \oplus \ldots \oplus (x_{in} \times y_{nj}) = \bigoplus_{n=1}^{m} x_{in} \otimes y_{nj} = \min_{n=1}^{m} (x_{in} + y_{nj})$$

The definition of the identity matrix in min-plus semirings needs to be addressed.

The identity of a matrix in algebra is a square matrix with the integer one in the diagonal and the integer zero in all other places where the column number and row number are not equal. Transforming an identity matrix into min-plus semirings occurs by replacing the diagonal with zero and all other values with infinity; the additive and multiplicative identities of the min-plus semirings.

If a problem consists of a matrix with more than ten columns or rows, the arithmetic can be quite lengthy and errors can occur, however, the computer science industry has constructed a couple of algorithms that can be used such as the Floyd-Warshall algorithm. For the purpose of this paper, the math will be done by hand to better show how to apply tropical algebra toward matrices. Using max-plus and min-plus semirings, this concept can be applied to the dequantization of non-linear equations to linear equations.

The Russian mathematician, Maslov, used semirings as a type of way to linearize a non-linear equation staying in the world of non-negative real numbers and notated it as $\mathbb{R}_+$. He defines a particular map, denoted as Φ_h, which represents the mapping of $\mathbb{R}_+$ to the max-plus semiring, $\mathbb{R} \cup \{-\infty\}$. He also does a change in variables where u maps to w and equals the product of a positive parameter (h) and the natural log of u, denoted as $u \to w = hlnu$ (Litvinov & Maslov, 1998). Maslov's research aided in the evolution of a theory known as the idempotent correspondence principle, which later leads to the subject of idempotent mathematics.

The idempotent correspondence principle is the existence of a "correspondence between important, interesting, and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings" (Litvinov, 2003, p.2). Maslov refers to this as the Maslov dequantization, which can be seen with the dequantization of the Hamilton-Jacobi equation starting from a simple heat equation shown below.

$$Heat\ Equation\colon \frac{\partial u}{\partial t} = \frac{h}{2} \times \frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$Where\ x \in \mathbb{R}, t > 0, and\ h\ is\ a\ positive\ parameter$$

$$Apply\ a\ change\ in\ variable\colon u \to w = -hlnu$$

$$u = e^{-\frac{w}{h}}$$

$$\frac{\partial u}{\partial t} = \left(U_w \times \frac{\partial w}{\partial t}\right) + \left(U_h \times \frac{\partial h}{\partial t}\right) \quad (2)$$

$$Because\ h\ is\ a\ parameter, the\ derivative\ of\ h\ with\ respect$$

$$to\ t\ is\ zero, equation\ (2)\ becomes$$

$$\frac{\partial u}{\partial t} = \left(U_w \times \frac{\partial w}{\partial t}\right)$$

$$\frac{\partial u}{\partial t} = \left(-\frac{e^{-\frac{w}{h}}}{h} \times \frac{\partial w}{\partial t}\right) \quad (3)$$

$$\frac{\partial^2 u}{\partial x^2} = \left(U_w \times \frac{\partial^2 w}{\partial x^2}\right) + \left(U_h \times \frac{\partial^2 h}{\partial x^2}\right) + \left(U_{ww} \times \left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$+ \left(2U_{wh}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial h}{\partial x}\right)\right) + \left(U_{hh}\left(\frac{\partial h}{\partial x}\right)^2\right) \quad (4)$$

$$\textit{Because h is a parameter, the derivative of h with respect to any variable is zero, equation } (4) \textit{ becomes the following}$$

$$\frac{\partial^2 u}{\partial x^2} = \left(U_w \times \frac{\partial^2 w}{\partial x^2}\right) + \left(U_{ww} \times \left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \left(-\frac{e^{-\frac{w}{h}}}{h} \times \frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{e^{-\frac{w}{h}}}{h^2} \times \left(\frac{\partial w}{\partial x}\right)^2\right) \quad (5)$$

$$\textit{Now plugging } (3) \textit{ and } (5) \textit{ in } (1)\textit{, the following is derived}$$

$$\left(-\frac{e^{-\frac{w}{h}}}{h} \times \frac{\partial w}{\partial t}\right) = \frac{h}{2} \times \left(-\frac{e^{-\frac{w}{h}}}{h} \times \frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{e^{-\frac{w}{h}}}{h^2} \times \left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$\frac{h}{e^{-\frac{w}{h}}}\left[\left(-\frac{e^{-\frac{w}{h}}}{h} \times \frac{\partial w}{\partial t}\right) = \left(-\frac{e^{-\frac{w}{h}}}{2} \times \frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{e^{-\frac{w}{h}}}{2h} \times \left(\frac{\partial w}{\partial x}\right)^2\right)\right]$$

$$\left(-\frac{\partial w}{\partial t}\right) = \left(\frac{-h}{2} \times \frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{1}{2} \times \left(\frac{\partial w}{\partial x}\right)^2\right) \quad (6)$$

According to Maslov, equation (6) is non-linear with w_1 and w_2 as solutions with the following properties $w_1 \oplus w_2 = hln(e^{\frac{w_1}{h}} + e^{\frac{w_2}{h}})$ and $w_1 \otimes w_2 = w_1 + w_2$ (Litvinov, 2007). During the conference in 2003, Litvinov defined that in equation (6), h goes to zero. The new equation follows:

$$\left(-\frac{\partial w}{\partial t}\right) = \left(\frac{1}{2} \times \left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$0 = \left(\frac{\partial w}{\partial t}\right) + \left(\frac{1}{2} \times \left(\frac{\partial w}{\partial x}\right)^2\right)$$

Equation 6 above has been transformed into a linear equation and uses the solutions of w_1 and w_2 under the max-plus semiring properties of tropical addition and multiplication.

CHAPTER V

APPLICATIONS

Application of Traditional Mathematics

The study of groups can be applied to molecular symmetry of molecules in chemistry. Molecular symmetry is used to study the patterns in structures and classify molecules under a certain symmetric label. Using molecular symmetry, one can predict the chirality of the molecule. Chirality refers to if a molecule can be superimposed on its mirror image; chiral means that a molecule cannot be superimposed on its mirror image and achiral means that it can (Atkins & de Paula, 2006) Figure 4 shows what it means for the compound to be chiral or achiral.

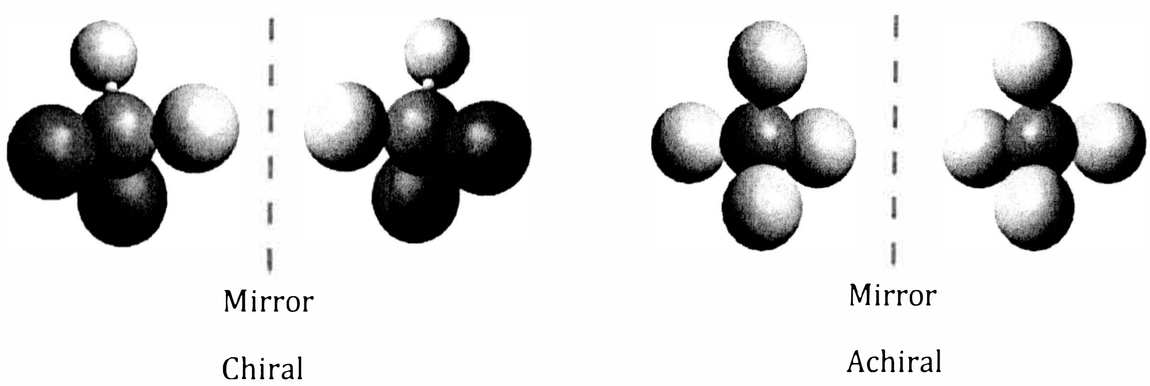


Figure 4: Chiral and Achiral pictorial representations

The type of symmetry present determines chirality; an achiral molecule has an improper rotation, which contains two successive transformations (Schonland, 1965). The symmetric elements will be described below along with how to apply molecular symmetry to group theory.

Using a water molecule as an example, the following will show how group theory applies to chemistry. Five elements of symmetry exist in the molecular symmetry: identity (E), n-fold axis of symmetry (C_n), reflection plane (σ), the inversion center (i) and the improper rotation axis (S_n). The identity element, denoted by E, can be explained when the molecule does not experience any transformation. The n-fold axis of symmetry leaves the molecule in an indistinguishable orientation after a rotation about the axis of 360°/n. Water has a 2-fold axis, denoted as C_2 because when rotating water 180°, the molecule looks to be the same, even though the hydrogen has switched positions (Schonland, 1965).

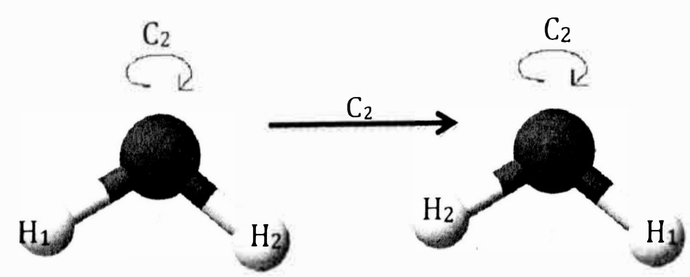


Figure 5: N-fold Configuration for Water

The C_2 rotation can be written into a matrix by showing the change in x, y and z coordinates. The origin of each coordinate system exists in the center of each atom. The column of the matrix represents the coordinates of the respective atoms as the following, x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3 and z_3. The rows of the matrix represent the new position of the coordinates with respect to each atom of water and are in the same order as the representation of the column. The matrix below shows the transformation C_2 being performed, where subscript x_1, y_1, z_1 refers to hydrogen one, x_2, y_2, z_2 refers to the oxygen and x_3, y_3, z_3 refers to hydrogen two.

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \xrightarrow{C_2} \begin{bmatrix} 0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ -1&0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0 \end{bmatrix}$$

The matrix above shows that hydrogen one and two switch positions and the x and y coordinates of all three atoms become negative while the z stays positive due to the rotation about the major z axis. Some molecules contain more than one axis of rotation; if that occurs the highest order axis can be referred to as the principal axis. The use of the principal axis will be referred to for the following symmetric element.

The reflection plane leaves the molecule in an indistinguishable orientation following a mirror reflection; there can be horizontal, vertical and dihedral reflection planes.

The vertical plane (σ_v) is parallel to the principal axis, the horizontal plane (σ_h) is perpendicular to the principal axis and the dihedral plane (σ_d) is a vertical mirror plane that bisects the angle between two C_2 axes. The vertical and dihedral planes are often confused with each other because they are both vertical, however there exists an easy way to distinguish the difference. The vertical planes go through the corners of the horizontal plane while the dihedral planes go through the midpoints of the sides of the horizontal plane (Atkins & de Paula, 2006). The diagrams are shown below in figure 6 for better clarification of how they correspond to the principal axis.

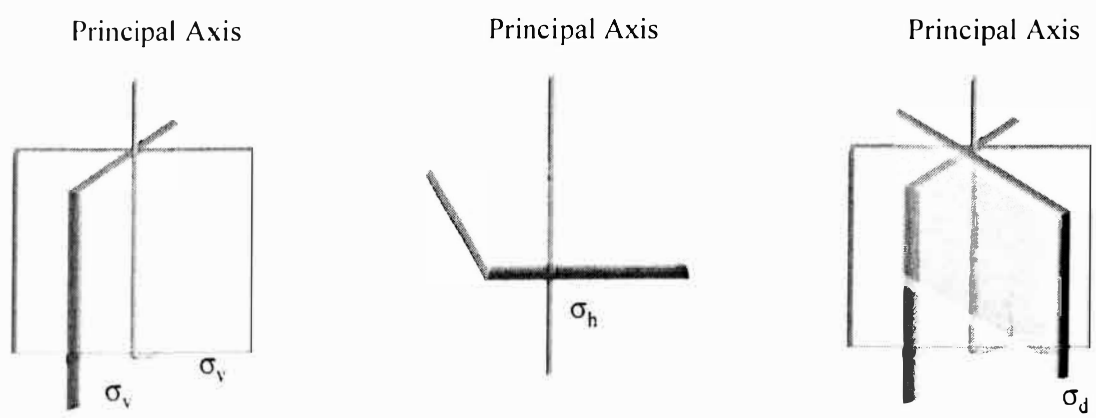


Figure 6: The three types of mirror planes

Water has two vertical reflection planes classifying water as C_{2v} because it has a 2-fold rotation as well as vertical reflection planes. The two planes go through the principal axis of rotation, one resides in the yz plane (σ_v) and the other in the xz plane (σ'_v), figure 7 summarizes the reflection planes.

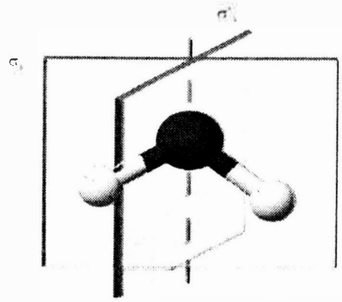


Figure 7: The mirror planes of Water

When going from the original orientation to σ_v, the x values become negative and the hydrogen stays in the same position. The matrix below shows the operation σ_v starting with the identity element.

$$\longrightarrow$$

When going from the identity to σ'_v, the y values become negative and the hydrogen changes position. The matrix becomes the following from the identity going to σ'_v.

$$\longrightarrow$$

The inversion center denoted by *i,* occurs when there exists a center O and the point (x, y, z) becomes (-x, -y, -z). A diagram below shows what happens to two points A and B under inversion.

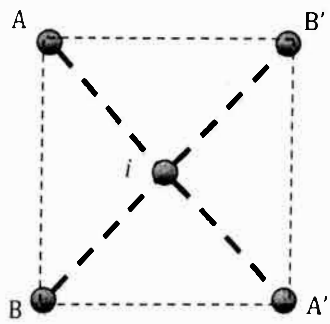


Figure 8: Pictorial Representation of an inversion center

A molecule has an inversion center if the operation produces an indistinguishable result (Atkins & de Paula, 2006). Using water as an example, it does not have an inversion center because when changing the position from H_1 to H_1', it does not produce an indistinguishable molecule.

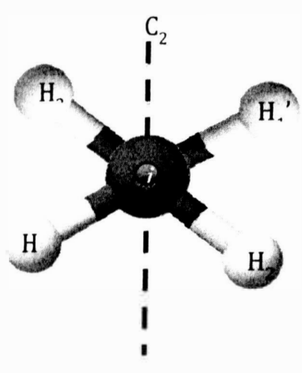


Figure 9: Water's inversion center

Although water does not have an inversion center, there are other molecules that do such as Benzene. Benzene, shown in figure 10 below, goes through an inversion process in which the molecule looks indistinguishable.

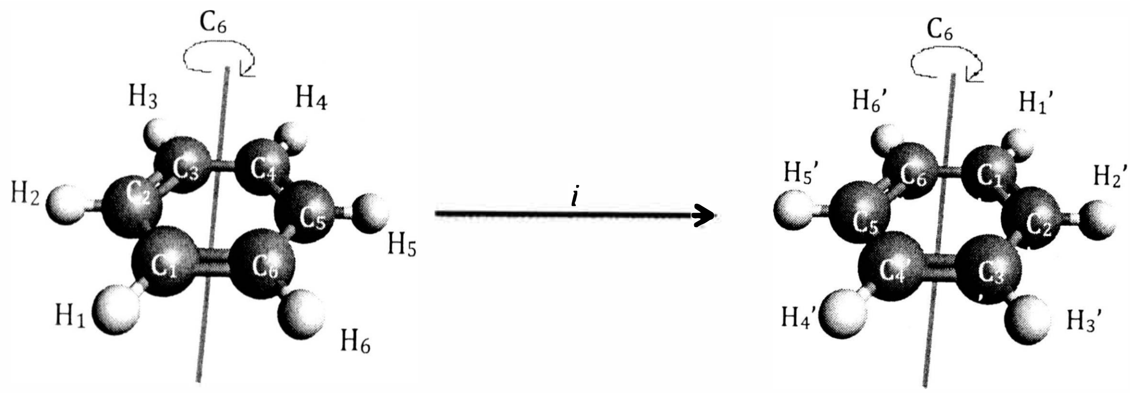


Figure 10: Benzene's inversion center

The last element of symmetry, the n-fold improper rotation axis denoted as S_n, consists of two successive transformation of two elements explained above whose combined operations produce an indistinguishable molecule. The transformation consists of a rotation of C_n and then a reflection σ_h (Schonland, 1965). Water does not contain the σ_h reflection plane for the operation to take place, however, benzene does contain a S_n operation with the rotation of C_6 and a horizontal reflection plane.

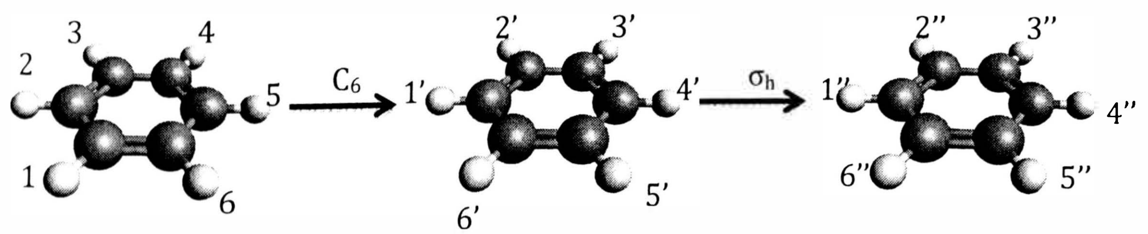


Figure 11: Benzene's improper rotation

Although water does not contain all of the elements of symmetry, group theory can still be applied to the molecule. To determine if the water molecule is a group, multiplication of the elements of molecular symmetry must be done to do so. To do multiplication, a table will be set up and filled in appropriately while doing the multiplication; the unfilled table in figure 12 follows.

*	E	C_2	σ_v	σ'_v
E				
C_2				
σ_v				
σ'_v				

Figure 12: Cayley Table

The first multiplication that needs to be completed is the identity with the other elements. Whenever the identity is multiplied by the element, the result is the element. Next will be the multiplication of C_2 with C_2, σ_v, and σ'_v.

$$C_2 * C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = E
$$

$$
C_2 * \sigma_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \sigma'_v
$$

$$
C_2 * \sigma'_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \sigma_v$$

For C_2, there is an identity element, an inverse and is closed under multiplication. So far, C_2 and E both satisfy the properties of a group. Next, to show that σ_v satisfies the properties of a group.

$$\sigma_v * C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \sigma'_v$$

$$\sigma_v * \sigma_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = E$$

$$\sigma_v * \sigma'_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C_2$$

For σ_v, there is an identity element, an inverse and is closed under multiplication. So far, σ_v, C_2 and E all satisfy the properties of a group. Next, to show that σ'_v satisfies the properties of a group.

$$\sigma'_v * C_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \sigma_v
$$

$$
\sigma'_v * \sigma_v = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C_2
$$

$$
\sigma'_v * \sigma'_v = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = E
$$

For σ'_v, there is an identity element, an inverse and is closed under multiplication. Therefore, σ'_v, σ_v, C_2 and E all satisfy the properties of a group. The Cayley table that goes with the operation, *, is shown in figure 13 below.

*	**E**	**C_2**	**σ_v**	**σ'_v**
E	E	C_2	σ_v	σ'_v
C_2	C_2	E	σ'_v	σ_v
σ_v	σ_v	σ'_v	E	C_2
σ'_v	σ'_v	σ_v	C_2	E

Figure 13: Complete Cayley Table

When determining if a molecule is a group, it must be closed under the operation being performed, contain an identity, be associative and contain an inverse. Looking at the table above, closure is satisfied because all the products are members of the group. For an identity to be satisfied, there must be an element E that when multiplied by a different element yields that element. To prove associativity, three elements are used and put into the form a*(b*c) = (a*b)*c, when performing all twenty-four combinations of three from 4 elements, associativity is satisfied. To determine if an inverse is present, every column or row must contain an identity element E. Looking at the table above, each column or row possesses an identity element therefore the inverse is satisfied. Water satisfies all properties of a group and can be considered a group under multiplication.

Idempotent Mathematics Application

An application of idempotent mathematics can be seen ın the shortest path problems. The shortest path problem uses the properties of the min-plus semiring to perform matrix algebra and matrix multiplication to find the shortest distance from point A to point H.

Figure 14 below shows the amount of time it took a mouse to go from one point to the other and back. The problem them is to figure out the shortest amount of time it takes to get from point A to point H and from point H to point A. To show how to solve this problem, matrix methods will be used.

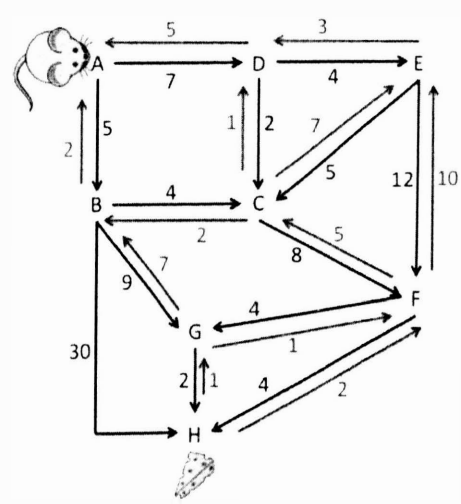


Figure 14: Maze

First, an adjacency matrix must be created that shows the amount of time to get from one point to another; if there is no travel to one point, infinity will be used to show there is no place holder for that destination. The column represents the starting point while the rows identify the path to another destination.

$$X = \begin{vmatrix} \infty & 5 & \infty & 7 & \infty & \infty & \infty & 30 \\ 2 & \infty & 4 & \infty & \infty & \infty & 9 & \infty \\ \infty & 2 & \infty & 1 & 7 & 8 & \infty & \infty \\ 5 & \infty & 2 & \infty & 4 & \infty & \infty & \infty \\ \infty & \infty & 5 & 3 & \infty & 12 & \infty & \infty \\ \infty & \infty & 5 & \infty & 10 & \infty & 4 & 4 \\ \infty & 7 & \infty & \infty & \infty & 1 & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty & 2 & 1 & \infty \end{vmatrix}$$

To find the shortest path time from point i to j, the following definition will be used

$$X^* = I \oplus X \oplus X^2 \oplus \ldots \oplus X^{n-1}$$ [Pan & Reif, n.d., p.501].

To obtain X^2, min-plus matrix multiplication must be performed to complete the problem. Under min-plus matrix multiplication, the product of two matrices takes the following notation:

$$(a_{i1} \otimes b_{1j}) \oplus (a_{i2} \otimes b_{2j}) \oplus \ldots \oplus (a_{in} \otimes b_{nj}) = \min_{k=1}^{n} (a_{ik} + b_{kj}).$$

Given the notation above, X^2 thru X^7 are calculated below:

$$X^2 = \begin{vmatrix} 7 & \infty & 9 & \infty & 11 & \infty & 14 & 35 \\ \infty & 6 & \infty & 5 & 11 & 10 & 31 & 11 \\ 4 & \infty & 3 & 10 & 5 & 19 & 11 & 12 \\ \infty & 4 & 9 & 3 & 9 & 10 & \infty & \infty \\ 8 & 7 & 5 & 6 & 7 & 13 & 16 & 16 \\ \infty & 7 & 15 & 6 & 12 & 5 & 5 & 6 \\ 9 & \infty & 6 & \infty & 11 & 4 & 3 & 5 \\ \infty & 8 & 7 & \infty & 12 & 2 & 6 & 3 \end{vmatrix}, \quad X^3 = \begin{vmatrix} \infty & 11 & 16 & 10 & 16 & 15 & 36 & 16 \\ 8 & 38 & 7 & 14 & 9 & 13 & 12 & 14 \\ 15 & 5 & 10 & 4 & 10 & 11 & 13 & 13 \\ 6 & 11 & 5 & 10 & 7 & 17 & 13 & 14 \\ 9 & 7 & 8 & 6 & 10 & 13 & 16 & 17 \\ 9 & 12 & 8 & 15 & 10 & 6 & 7 & 7 \\ \infty & 8 & 9 & 7 & 13 & 4 & 6 & 5 \\ 10 & 9 & 7 & 8 & 12 & 5 & 4 & 6 \end{vmatrix},$$

$$X^4 = \begin{vmatrix} 13 & 18 & 12 & 17 & 14 & 18 & 17 & 19 \\ 19 & 9 & 14 & 8 & 14 & 13 & 15 & 14 \\ 7 & 12 & 6 & 11 & 8 & 14 & 14 & 15 \\ 13 & 7 & 12 & 6 & 12 & 13 & 15 & 15 \\ 9 & 10 & 8 & 9 & 10 & 16 & 16 & 17 \\ 14 & 10 & 11 & 9 & 15 & 8 & 8 & 9 \\ 10 & 11 & 9 & 10 & 11 & 7 & 6 & 8 \\ 11 & 9 & 10 & 8 & 12 & 5 & 7 & 6 \end{vmatrix}, \quad X^5 = \begin{vmatrix} 20 & 14 & 19 & 13 & 19 & 18 & 20 & 19 \\ 11 & 16 & 10 & 15 & 12 & 16 & 15 & 17 \\ 14 & 8 & 13 & 7 & 13 & 14 & 16 & 16 \\ 9 & 14 & 8 & 13 & 10 & 16 & 16 & 17 \\ 12 & 10 & 11 & 9 & 13 & 16 & 18 & 18 \\ 12 & 13 & 11 & 12 & 13 & 9 & 10 & 10 \\ 13 & 11 & 12 & 10 & 14 & 7 & 9 & 8 \\ 11 & 12 & 10 & 11 & 12 & 8 & 7 & 9 \end{vmatrix},$$

$$X^6 = \begin{bmatrix} 16 & 21 & 15 & 20 & 17 & 21 & 20 & 22 \\ 18 & 12 & 17 & 11 & 17 & 16 & 18 & 17 \\ 10 & 15 & 9 & 14 & 11 & 17 & 17 & 18 \\ 16 & 10 & 15 & 9 & 15 & 16 & 18 & 18 \\ 12 & 13 & 11 & 12 & 13 & 19 & 19 & 20 \\ 15 & 13 & 14 & 12 & 16 & 11 & 11 & 12 \\ 13 & 14 & 12 & 13 & 14 & 10 & 9 & 11 \\ 14 & 12 & 13 & 11 & 15 & 8 & 10 & 9 \end{bmatrix}, \quad X^7 = \begin{bmatrix} 23 & 17 & 22 & 16 & 22 & 21 & 23 & 22 \\ 14 & 19 & 13 & 18 & 15 & 19 & 18 & 20 \\ 17 & 11 & 16 & 10 & 16 & 17 & 19 & 19 \\ 12 & 17 & 11 & 16 & 13 & 19 & 19 & 20 \\ 15 & 13 & 14 & 12 & 16 & 19 & 21 & 21 \\ 15 & 16 & 14 & 15 & 16 & 12 & 13 & 13 \\ 16 & 14 & 15 & 13 & 17 & 10 & 12 & 11 \\ 14 & 15 & 13 & 14 & 15 & 11 & 10 & 12 \end{bmatrix}.$$

Using X through X^7, X^* can be calculated using the definition of min-plus matrix addition:

$$X^* = I \oplus X \oplus X^2 \oplus X^3 \oplus X^4 \oplus X^5 \oplus X^6 \oplus X^7 = \begin{bmatrix} 0 & 5 & 9 & 7 & 11 & 15 & 14 & 16 \\ 2 & 0 & 4 & 5 & 9 & 10 & 9 & 11 \\ 4 & 2 & 0 & 1 & 5 & 8 & 11 & 12 \\ 5 & 4 & 2 & 0 & 4 & 10 & 13 & 14 \\ 8 & 7 & 5 & 3 & 0 & 12 & 16 & 16 \\ 9 & 7 & 5 & 6 & 10 & 0 & 4 & 4 \\ 9 & 7 & 6 & 7 & 11 & 1 & 0 & 2 \\ 10 & 8 & 7 & 8 & 12 & 2 & 1 & 0 \end{bmatrix}$$

Reading the matrix above, the shortest time between the mouse and the cheese is 16 seconds going from A to B, B to G and G to H. If the mouse went back to point A from point H, the minimum time would be 10 seconds going from H to G, G to B and B to A. The use of min-plus semiring is applied to many network problems and the calculations are usually done through algorithms.

A Theoretical Relationship Through Experimentation

The purpose of this thesis was to find a connection between mechanics and mathematics theoretically instead of just having an analogous behavior described by Maslov and Litvinov. Figure 15 below shows the idea that Maslov and Litvinov described in the 2003 conference in Vienna, Austria. To show the validity of the figure, the connection between traditional mathematics and idempotent mathematics was described above to connect the two using idempotent correspondence principle also known as Maslov's dequantization. One example includes the linearization of the heat equation yielding the Hamilton-Jacobi equation under the conditions of a max-plus semiring.

When showing the validity of how quantum mechanics and classical mechanics are related to each other, an experiment known as the particle in the box experiment was conducted to show that Neil Bohr's correspondence principle connects classical and quantum mechanics.

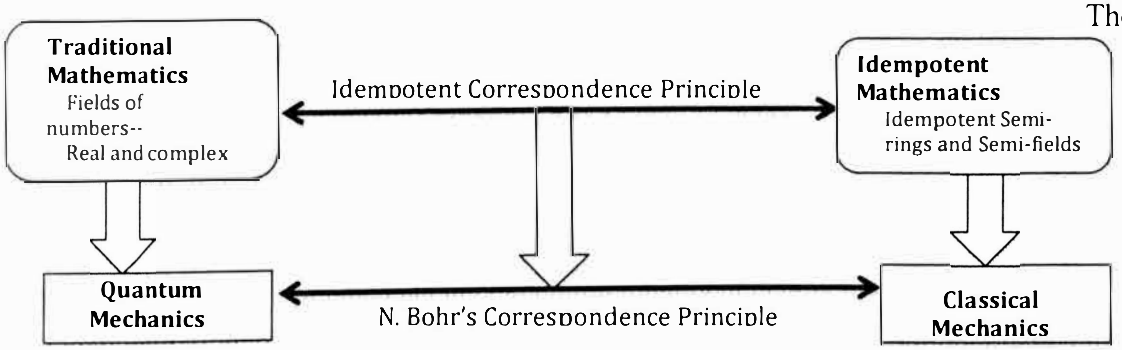


Figure 15: Analogous behavior between mathematics and mechanics.

The experiment used three organic solids, 1,4-diphenyl-1,3-butadiene, 1,6-diphenyl-1,3,5-hexatriene and 1,8-diphenyl-1,3,5,7-octatetraene shown in figure 16 below. The figure shows the name, structure, molecular weight and density of each solid, which are later used in determining the concentration of dilutions.

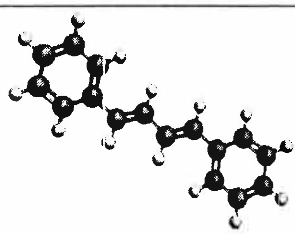
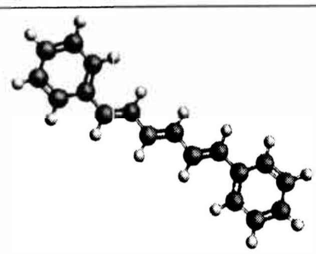
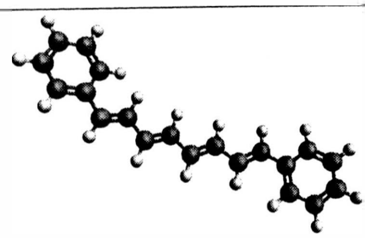
Chemical Name	Structure	Molecular Weight	Density
1,4-diphenyl-1,3-butadiene		206.282 grams per mole	1.035 grams per milliliter
1,6-diphenyl-1,3,5-hexatriene		232.320 grams per mole	1.028 grams per milliliter
1,8-diphenyl-1,3,5,7-octatetraene		258.357 grams per mole	1.023 grams per milliliter

Figure 16: Compound names, structures, molecular weights and density used in Particle in box calculations.

Each solid was separately diluted in Cyclohexane to produce samples with concentrations of approximately 10^{-6} Molar. To get 10^{-6} Molar, dilutions had to be performed; tables 1, 2 and 3 below show the dilution.

Beaker	Starting Amount	Ending Amount	Concentration
1	0.001 grams $C_{16}H_{14}$ + 10 mL C_6H_{12}	10 mL solution	5×10^{-4} M
2	1 mL Beaker 1 + 4 mL C_6H_{12}	5 mL solution	1×10^{-4} M
3	1 mL Beaker 2 + 4 mL C_6H_{12}	5 mL solution	2×10^{-5} M
4	1 mL Beaker 3 + 4 mL C_6H_{12}	5 mL solution	4×10^{-6} M
5	1 mL Beaker 4 + 3 mL C_6H_{12}	4 mL solution	1×10^{-6} M

Table 1: The dilutions of 1,4-diphenyl-1,3-butadiene

Beaker	Starting Amount	Ending Amount	Concentration
1	0.001 grams $C_{18}H_{16}$ + 10 mL C_6H_{12}	10 mL solution	4 x 10^{-4} M
2	1 mL Beaker 1 + 4 mL C_6H_{12}	5 mL solution	8 x 10^{-5} M
3	1 mL Beaker 2 + 4 mL C_6H_{12}	5 mL solution	2 x 10^{-5} M
4	1 mL Beaker 3 + 4 mL C_6H_{12}	5 mL solution	4 x 10^{-6} M
5	1 mL Beaker 4 + 3 mL C_6H_{12}	4 mL solution	1 x 10^{-6} M

Table 2: The dilutions of 1,6-diphenyl-1,3,5-hexatriene

Beaker	Starting Amount	Ending Amount	Concentration
1	0.001 grams $C_{20}H_{16}$ + 10 mL C_6H_{12}	10 mL solution	4 x 10^{-4} M
2	1 mL Beaker 1 + 4 mL C_6H_{12}	5 mL solution	8 x 10^{-5} M
3	1 mL Beaker 2 + 4 mL C_6H_{12}	5 mL solution	2 x 10^{-5} M
4	1 mL Beaker 3 + 4 mL C_6H_{12}	5 mL solution	4 x 10^{-6} M
5	1 mL Beaker 4 + 3 mL C_6H_{12}	4 mL solution	1 x 10^{-6} M

Table 3: The dilutions of 1,8-diphenyl-1,3,5,7-octatetraene

Once the 10^{-6} Molar concentrations were made, the solutions were placed into a UV-visible spectrophotometer to measure the absorbance of each sample compound at a visible wavelength. The purpose of using the UV-visible spectrophotometer is to find the maximum wavelength, known as lambda max denoted λ_{max}, and compare it to the theoretical λ_{max}. When the samples were placed into the UV-visible spectrophotometer, the experimental λ_{max} for 1,4-diphenyl-1,3-butadiene, 1,6-diphenyl-1,3,5-hexatriene and 1,8-diphenyl-1,3,5,7-octatetraene were 295 nm, 310 nm and 340 nm respectively. These values were compared to the theoretical maximum wavelengths.

To get the theoretical maximum wavelengths, the box lengths needed to be calculated. Using Avogadro, a 3D molecular construction software, the theoretical box lengths were calculated for 1,4-diphenyl-1,3-butadiene, 1,6-diphenyl-1,3,5-hexatriene and 1,8-diphenyl-1,3,5,7-octatetrane to be 6.34 Angstroms, 8.793 Angstroms and 11.251 Angstroms respectively.

The box lengths represent the distance between the phenyl groups using an approximated bond length for each carbon to carbon double bond. With the summation of bond lengths, the theoretical box lengths can be placed into the combination of the Energy Level equation (a) and the Photon Energy equation (b) obtaining a formula for lambda max (c):

$$\Delta E = \frac{n^2 h^2}{8mL^2} \qquad (a)$$

$$\lambda = \frac{hc}{\Delta E} \qquad (b)$$

$$\lambda = \frac{8mcL^2}{hn^2} \qquad (c)$$

Where *m* represents the mass of an electron, 9.109 x 10^{-31} kilograms, *c* represents the speed of light, 2.998 x 10^8 meters per second, *L* represents the theoretical box length in meters, *h* represents Planck's constant, 6.626 x 10^{-34} Joules second, and *n* represents the difference in energy state (Atkins and de Paula, 2006).

Inserting the aforementioned mathematical values into (c), the theoretical lambda maxes obtained for 1,4-diphenyl-1,3-butadiene, 1,6-diphenyl-1,3,5-hexatriene and 1,8-diphenyl-1,3,5,7-octatetraene were 265 nanometers, 283 nanometers and 321 nanometers respectively.

The purpose of finding the maximum wavelength was to show how the wave function changes as the quantum number, n, increases or decreases. A quantum number "is an integer…that labels the state of the system" and is "used to calculate the energy corresponding to the state" (Atkins and de Paula, 2006, p. 280). Based on this information, quantum mechanics and classical mechanics in terms of probability density and energy quantization will become consistent with large quantum numbers. When looking at the theoretical and experimental values for λ_{max}, there arose the question of error in the experiment or calculations. The experiment was performed again and received relatively the same results. When trying to determine why the numbers were off, the question arose about which of Schrödinger's equation was going to be used to try to relate the experiment to idempotent analysis. During the process of determining if the time independent or time dependent formula was to be used to try to bring it back to the Hamilton-Jacobi equation, there was error in deriving the correct formula. Due to the complexity of these collected ideas, the original goal was not obtained. In future research, the particle in a box experiment and more theoretical research will be used to attempt to find the existence of such a connection and to find equations that directly relate mathematics to mechanics.

CHAPTER VI

CONCLUSION

Due to centuries of research and discovery, mathematics has evolved from the process of solving equations arithmetically to theoretically. The Egyptians and Babylonians contributed to the foundation of mathematics through the types of arithmetic done such as the doubling method. The development of their ideas during that era aided Greek and Islamic mathematicians with the discovery of geometry along with algebra. During the creation of geometry and algebra, questions began to arise of why equations work the way they do, causing a transition into a new era of mathematics.

Traditional mathematics started because of the need to know why something worked for one solution but not the other. This evolved into theories such as group, ring and field theory, known today as abstract algebra. These theories contributed to the knowledge of properties for certain number systems as well as the application to fields such as chemistry and mechanics. Group theory has more of a sound foundation when it comes to applications in chemistry and cryptography, but field theory and ring theory are slowly emerging into more subjects other than mathematics that are still being researched. The theories explained in chapter three contain subcategories, which add more restrictions to the general definition such as semirings.

Semirings have been applied to computer science in areas such as automata language and algorithms such as the one used for the shortest path problem. The max-plus and min-plus semirings contribute to the answer of the shortest path problem as well as to the linearization of a non-linear equation making it simpler to solve. The idea of the max-plus and min-plus semirings helped with the beginning of idempotent mathematics due to Masolv and Litvinov.

According to Litvinov and Maslov, "(t)here is a correspondence between...the field of real (or complex) numbers and similar constructions and results over idempotent semirings in the spirit of (Neil) Bohr's correspondence principle in Quantum Mechanics" (Litvinov, 2003, p.2). Neil Bohr's correspondence principle "is the notion that quantum mechanics must resemble classical behavior for large energies" and contains several rules that explain how to represent that behavior (Litvinov, 2007). The idea of the analogous behavior between the two subjects will be continued as further research where the connections between the subjects will be found through proofs and experiments such as the experiments performed by Erwin Schrödinger and Niels Bohr.

The continuation of this research will hopefully contribute to finding probabilistic outcomes of diseases in the medical field. Min-plus semirings have just made a step into genetic coding and how it will form based on the DNA from a set of parents. The purpose of future research in this field is to apply the shortest path problem to genetics and use the Markov chain to determine the probability of retrieving a disease based on genetics and medical history.

REFERENCES

Algebra. (2013). In *Encyclopædia Britannica*. Retrieved from http://www.britannica.com/EBchecked/topic/14885/algebra/231065/Islamic-contributions.

Atkins , P., & de Paula, J. (2006). *Physical chemistry*. (8th edition ed., pp. 270-283). Great Britain: Oxford University Press.

Bronson, R., & Costa, G. B. (2009). *Matrix methods: Applied linear algebra*. (3rd edition ed.). Burlington: Academic Press.

Clark, A. (1984). *Elements of abstract algebra*. (pp. 17-160). New York: Dover Publications Inc.

Cohen, G., Gaubert, S., Quadrat, J., & Singer, I. (2003). Max-plus convex sets and functions. In G. Livinov & V. Maslov (Eds.), *Idempotent Mathematics and Mathematical Physics* (Vol. 377, pp. 109-129). Providence: American Mathematical Society.

Corry, L. (1998-2000). The origins of the definition of abstract rings. *Modern Logic*, *8.1*(2), 5-27. Retrieved from http://projecteuclid.org/DPubS/Repository/1.0/Disseminate?view=body&id=pdf_1&handle=euclid.rml/1081878062.

Ellis, A. (Nov. 2005). *Tropical algebra.* Retrieved from http://www.math.utah.edu/mathcircle/notes/MathCircleIv2.pdf.

Farlow, K. (2009). *Max-plus algebra.* (Master's thesis, Virginia Polytechnic Institute and State University)Retrieved from scholar.lib.vt.edu/theses/available/etd../Finalthesis.pdf

Glazek, K. (2002). *A guide to the literature on semirings and their applications in mathematics and information sciences: with complete bibliography*. Boston: Kluwer Academic Publishers.

Golan, J. S. (2005, March). *Some recent applications of semiring theory.* Retrieved from http://moonstone.math.ncku.edu.tw/AlgebraConference/golantalk.pdf

Hagenauer, J., & Hoeher, P. (1989, Nov.). *A viterbi algorithm with soft-decision outputs and its applications*. Ieee global telecommunications conference, Dallas, Texas. Retrieved from http://www-ict.tf.uni-kiel.de/download

Hebisch, U., & Weinert, H. J. (1998). *Semirings: Algebraic theory and applications in computer science*. Singapore: World Scientific.

Idempotent. 2012. In *Merriam-Webster.com*.Retrieved from http://www.merriam-webster.com/dictionary/idempotent.

Invariant. (n.d.) *The American Heritage® Dictionary of the English Language, Fourth Edition*. (2003). Retrieved March 25 2013 from http://www.thefreedictionary.com/invariant.

Kleiner, I. (1986, Oct.). The evolution of group theory: a brief survey. *Mathematical Association of America*, *59*(4), 195-215. Retrieved from http://www.jstor.org/stable/2690312.

Kleiner, I. (2007). *A history of abstract algebra.* (pp. 1-78). Boston: Birkhäuser.

Kouba. (n.d.). *Find the second partial derivative using the chain rule.* Retrieved from http://www.math.ucdavis.edu/~kouba/Math21CHWDIRECTORY/SecondPartial.pdf.

Lewinter, M., & Widulski, W. (2002). *The saga of mathematics: A brief history*. Upper Saddle River: Prentice-Hall Inc.

Lidl, R., & Pilz, G. (1984). *Applied abstract algebra*. New York: Springer-Verlag.

Litvinov, G. L. (2007). Maslov dequantization, idempotent and tropical mathematics: a brief introduction. *Journal of Mathematical Sciences*, *140*(3), 426-429.

Litvinov, G. L., & Maslov, V. P. (1998). The correspondence principle for idempotent calculus and some computer applications. In J. Gunawardena (Ed.), *Idempotency* (pp. 420-443). Cambridge University Press.

Mackiw, G. (1985). *Applications of abstract algebra*. New York: Wiley.

Mathematical proof. (n.d.) *WordNet 3.0, Farlex clipart collection*. (2003-2008). Retrieved from http://www.thefreedictionary.com/mathematical+proof.

Nicholson, W. K. (2007). *Introduction of abstract algebra*. (3rd Edition ed., pp. 11-12,73-103,155-174). Hoboken: John Wiley & Sons, Inc.

Nicodemi, O., Sutherland, M. A., & Towsley, G. W. (2007). *An introduction to abstract algebra, with notes to the future teacher*. (pp. 82-224). Upper Saddle River: Prentice Hall.

O'Connor, J. J., & Robertson, E. F. (Dec. 2000). *Babylonian numerals*. Retrieved from http://www-history.mcs.st-and.ac.uk/HistTopics/Babylonian numerals.html.

O'Connor, J. J., & Robertson, E. F. (Dec. 2000). *Egyptian numerals*. Retrieved from http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian numerals.html.

Pan, V., & Reif, J. (1989). Fast and efficient solution of path algebra problems. *Journal of Computer and System Sciences*, *38*(3), 494-510. Retrieved from http://www.cs.duke.edu/~reif/paper/pan/PNDpaths.pub.pdf

Peacock, G. (1830). *A treatise on algebra*. London: J. & J.J. Deighton.

Pin, J. (n.d.). *Tropical semirings*. Retrieved from http://liafa.jussieu.fr/web9/rapportrech/rapportsWeb/1995/Tropical.pdf

Rowland, Todd. "Algebraic Geometry." From *MathWorld*--A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/AlgebraicGeometry.html.

Schonland, D. S. (1965). *Molecular symmetry: an introduction to group theory and its uses in chemistry*. (1st edition ed.). Princeton: D. Van Nostrand.

Sen, M. K., & Adhikari, M. R. (1993). On maximal k-ideals of semirings. *Proceedings of the American Mathematical Society*, *118*(3), 699-703. Retrieved from http://www.jstor.org/stable/2160106

Weisstein, Eric W. "Fundamental Theorem of Algebra." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/FundamentalTheoremofAlgebra.html.

Weisstein, E. W. (n.d.). "Quaternion." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Quaternion.html.