RINGS OF QUOTIENTS AND LOCALIZATION

A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF MASTER OF ARTS IN MATHEMATICS

IN THE GRADUATE SCHOOL OF THE

TEXAS WOMAN'S UNIVERSITY

COLLEGE OF

ARTS AND SCIENCES

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DENTON, TEXAS

MAY, 1974

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INTRODUCTION

The concept of a ring of quotients was apparently first introduced in 1927 by a German mathematician Heinrich Grell in his paper "Bezeihungen zwischen Ideale verschievener Ringe" [7]. In his work Grell observed that it is possible to associate a ring of quotients with the set S of non-zero divisors in a ring. The elements of this ring of quotients are fractions whose denominators belong to S and whose numerators belong to the commutative ring. Grell's ring of quotients is now called the classical ring of quotients. [1]

Grell's concept of a ring of quotients remained virtually unchanged until 1944 when the Frenchman Claude Chevalley presented his paper, "On the notion of the Ring of Quotients of a Prime Ideal" [5]. Chevalley extended Grell's notion to the case where S is the complement of a prime ideal. (Note that the set of all non-zero divisors and the set-theoretic complement of a prime ideal are both instances of multiplicative sets -- sets that are closed under multiplication.)

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[1] According to V. P. Elizarov, the Russian mathematician E. Steinitz introduced the concept of quotient rings in 1909 in his paper "The Algebraic Theory of Rings," but this paper is not available for inspection. See V. P. Elizarov, "Rings of Quotients," Algebra and Logic, 8(1969), 219.

Chevalley confined his applications to Noetherian Rings. A ring of quotients associated with the complement of a prime ideal P is sometimes called the localization of R at P.

Four years later, the Russian A. I. Uzkov generalized completely the concept of rings of quotients in his paper "On Rings of Quotients of Commutative Rings" [11]. Uzkov showed that it is possible to construct a ring of quotients from an arbitrary commutative ring with an arbitrary multiplicative set. Uzkov's ring of quotients is now referred to as the general ring of quotients.

Since 1927 when it was first introduced, the concept of rings of quotients has become a unifying idea in commutative ring theory and thus in commutative algebra. Chevalley appears to have been led to the idea of rings of quotients more because of its "usefulness in applications to algebraic geometry, than because of the important role it can be made to play in the abstract theory" [10, p. 103]. The process of localizing a ring at a prime ideal is the algebraic analogue of concentrating attention on neighborhoods of points in geometry. Thus the "results about localization can usefully be thought of in geometric terms" [1, p. vii]. As a result, "commutative algebra is now one of the foundation stones of algebraic geometry. It provides the complete local tools for the subject . . . " [1, p. vii]. The modern trend in most all areas of algebra puts more emphasis on modules and localization, so

that the rings of quotients have applications in many areas of algebra. Probably the main reason for this trend is that the passage to the rings of quotients makes many proofs shorter and considerably easier.

Many of the more important and interesting properties of rings of quotients are now so well known to the experienced mathematician that their proofs are often left to the reader. Indeed, in no single source are all the proofs available together. Such scholars as Kaplansky and Northcutt present some proofs, but in every case some of the relevant properties are merely assumed and not fully developed. The purposes of this paper, therefore, are to give a detailed construction of the ring of quotients deriving some of the well-known properties, and to discover which properties of the ring R are preserved under localization. Most of the theorems and problems in this paper are taken from Kaplansky's Commutative Rings and Gilmer's Multiplicative Ideal Theory.

The first chapter of this paper will include a detailed study of multiplicative sets and prime ideals. The second chapter will include the properties of the generalized ring of quotients, while the third chapter will examine some of the properties of the classical ring of quotients. The last chapter will be devoted to a consideration of properties of rings that are preserved under localization.

CHAPTER I

PRIME IDEALS AND MULTIPLICATIVE SETS

Prime ideals not only play a central role in the theory of commutative rings, but they also play an important role in the process of localization. Indeed Chevalley defined his multiplicative sets as the set-theoretic complements of prime ideals. Hence it seems appropriate to devote the first part of this paper to a few observations concerning prime ideals and to some of the properties of multiplicative sets. Unless explicitly stated to the contrary, in all that follows R will denote a commutative ring with an identity and S will be used for any multiplicative set.

An ideal P in a ring is prime if ab in P implies either a or b is in P. A subset S of R is a multiplicative set if S contains the identity and 1) for a and b in S, the product ab is in S and 2) the zero element is not in S. Examples of multiplicative sets include the following.

1. The set consisting of just the unity element is trivially a multiplicative set.

2. The set S of all non-zero divisors is a multiplicative set. For if a and b are in S then $ab \neq 0$ since $a \neq 0$ and $b \neq 0$. If $0 = (ab)c = a(bc)$ then $c = 0$ since a and b are in S, and hence ab is in S.

3. The set S of invertible elements in R is a multiplicative set.

For if we let a and b be elements of S, then a and b are invertible. Thus $1 = (ab)b^{-1}a^{-1} = (ab)(ab)^{-1}$. Therefore ab is invertible and so is an element of S.

4. The set S of all powers of a non-nilpotent element is a multiplicative set. If $\{1, a^1, a^2, \ldots, a^n, \ldots\}$ is the set of all powers of a, then $a^k \cdot a^\ell = a^{k+\ell}$ is in S since a is non-nilpotent.

5. The set S of all integers not divisible by an arbitrary fixed prime is a multiplicative set. For if m and n are elements of S, then p does not divide m or n. Thus p does not divide mn. Therefore mn is in S.

6. The set-theoretic complement of a prime ideal is a multiplicative set. The proof that this set is a multiplicative set is contained in Theorem 1, which also gives us another characterization of prime ideals.

Theorem 1. Let I be an ideal, and let S be the set-theoretic complement of I. Then I is prime if and only if S is a multiplicative set.

Proof: Suppose S is not a multiplicative set. Then for some a and b in S, ab is not in S. Thus ab is in I, and I is not prime. Now suppose I is not prime. Then for some ab in I, neither a nor b is in I. Thus a and b are in S and S is not a multiplicative set.

The ideal I in Theorem 1 is obviously maximal with respect to the exclusion of S. Actually the weaker condition that an ideal I need only

be maximal with respect to the exclusion of S is a sufficient condition for I to be prime.

Theorem 2. Let S be a multiplicative set in a ring R. An ideal I in R is prime if I is maximal with respect to the exclusion of S.

Proof: Suppose ab is in I and neither a nor b is in I. Then the ideal (I, a) is strictly larger than I. Therefore (I, a) intersects S. Thus there exists an s' in S of the form $s' = i' + xa$ (i' in I, x in R). Similarly we have s'' in S and $s'' = i'' + yb$. But then

$$s's'' = (i' + xa)(i'' + yb) = i'i'' + i'by + i''xa + xyab.$$

Now i'i'' is in I obviously, i'yb is in I since i is in I and yb is in R. Similarly i''xa is in I. Also xyab is in I since ab is in I. Therefore s's'' is in I. But this is a contradiction, since $S \cap I = \emptyset$.

This theorem suggests a method for constructing prime ideals from ordinary ideals in the ring R.

Theorem 3. Let S be a multiplicative set. If an ideal J of R does not meet S, then there is a maximal ideal I with respect to the exclusion of S such that I contains J. Such an ideal I is prime.

Proof: Suppose J is not maximal, and let a be an element in neither J nor S. Then (J, a), the ideal generated by J and a properly contains J. If (J, a) is maximal, then (J, a) is the required ideal. If (J, a) is not maximal, then there exists an element b in neither (J, a) nor S such that (J, a, b) properly contains (J, a). Now (J, a, b) is either

maximal or not maximal, and we repeat the argument above. We thus obtain an inductive set. Hence, Zorn's lemma implies the existence of I. By theorem 2, I is prime.

Not only can we find maximal ideals with respect to the exclusion of S, but we can also find maximal multiplicative sets with respect to the exclusion of a given ideal I. This result is stated as Theorem 4. A proof can be found in McCoy [9, p. 104].

Theorem 4. Let I be an ideal in a commutative ring R and S a multiplicative set of R which does not meet I. Then S is contained in a maximal multiplicative set T which does not meet I; that is, if M is a multiplicative set such that T is a proper subset of M, then M contains an element of I.

If a multiplicative set S is the complement of a prime ideal, then it has the additional property of being saturated. That is, if an element x is in S, then all divisors of x are also in S. All the examples on pages 4 and 5 are saturated except for example 1. That the set of all invertible elements is a saturated multiplicative set is immediate. For if ab is in S, then $(ab)^{-1} = b^{-1} a^{-1}$ exists. Thus a and b are invertible and in S. The proofs of the other examples are just as trivial and will be omitted. Other examples of saturated multiplicative sets include the following.

1. If R is an integral domain and S is the set of all elements

expressible as products of principal primes, then S is a saturated multiplicative set. An element p is a principal prime if the ideal (p) is prime and non-zero. This set is obviously closed. The proof that S is saturated can be found in Kaplansky [8, p. 4].

2. The set S is a saturated multiplicative set if and only if the complement of S is the set-theoretic union of prime ideals in R. Let S be a saturated multiplicative set, and let T be the union of all prime ideals that do not intersect S. If an element a is in the complement of S, then the principal ideal (a) is disjoint from S since S is saturated. If we expand (a) to an ideal I maximal with respect to the exclusion of S, then I is prime. Thus every a not in S is in a prime ideal disjoint from S, so the complement of S is a subset of the union of prime ideals in R. The reverse inclusion follows from the definition of T, and so $S' = T$. Now suppose the complement of S is the set-theoretic union of prime ideals in R. The proof that S is multiplicative is similar to the proof of theorem 1, so will be omitted. If ab is in S, then ab is not an element of any prime ideal P of R. Thus a is not in P and b is not in P. Therefore both a and b are in S, so S is saturated.

It follows from example 2 that the set of zero divisors in R is a union of prime ideals. A saturated multiplicative set containing 0 is the whole ring, for if 0 is in S, then $a \cdot 0 = 0$ is in S for all a in R. Therefore a is in S since S is saturated.

CHAPTER II

RINGS OF QUOTIENTS

In this chapter and the next we shall make a systematic investigation of the properties of rings of quotients. We will consider the most general case in this chapter, where S is any multiplicative set, then restrict S to the set of non-zero divisors in Chapter 3.

Let us now consider the set of all symbols of the form (r, s) with r in R and s in S. We say (r, s) is equivalent to (r', s'), denoted by $(r, s) \sim (r', s')$ provided $t(s'r - sr') = 0$ for some t in S. We will show that $\sim$ is an equivalence relation. To prove that $\sim$ is reflexive, let t be an element of S. Then $t(0) = t(as - sa) = 0$, and so $(a, s) \sim (a, s)$. The relation is symmetric for if $(a, s) \sim (r, t)$ then

$$x(at - sr) = 0 = xat - xsr = xta - xsr = xsr - xta = x(rs - ta)$$

for some x in S. Thus $(r, t) \sim (a, s)$. The relation is transitive since if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $r(da - bc) = rda - rbc = 0$, and $t(fc - de) = tfc - tde = 0$, for some r and t in S. Thus $rda = rbc$ and $tfc = tde$. Multiplying the first equation by tf and the second by rb we get $trdaf = trbcf$ and $rtfcb = trdeb$. Thus $trdaf = trdeb$ and

$$trdaf - trdeb = trd(af - eb) = 0.$$

Therefore $(a, b) \sim (e, f)$ since trd is in S. The notation for the equivalence

class containing (r, s) will be r/s. The process of forming equivalence classes is called a localization of R by S.

If multiplication and addition are defined by analogy with the operations for ordinary fractions, the collection of all equivalence classes forms a ring, called the ring of quotients of R by S and denoted by $S^{-1}R$. That is, $S^{-1}R = \{r/s \mid r \text{ is in } R,\ s \text{ in } S\}$. Multiplication and addition are defined as follows:

$$(a/s)(b/t) = (ab)/(st) \text{ and } a/s + b/t = (ta + sb)/st.$$

These operations are well defined. Suppose $a/s = a'/s'$ and $b/t = b'/t'$. We need to show that $(ab)/(st) = (a'b')/(s't')$. Since $a/s = a'/s'$ and $b/t = b'/t'$ then $x(s'a - a's) = 0$ and $y(t'b - tb') = 0$ for some x and y in S. Hence $xs'a = xa's$ and $yt'b = ytb'$ and so $xs'ayt'b = xa'sytb'$. Thus

$$xs'yat'b - xa'sytb' = xy(s'at'b - a'stb') = 0.$$

Therefore $(ab)/(st) = (a'b')/(s't')$. To show that addition is well defined we must show that $(at + sb)/(st) = (a't' + s'b')/(s't')$. Multiplying $x(s'a - a's)$ and $y(t'b - tb')$ by $yt't$ and $xs's$ respectively, we have

$$xy(t'ts'a - t'ta's) = 0 = xy(s'st'b - s'stb').$$

Thus $xy(t'ts'a - t'ta's + s'st'b - s'stb') = xy[t's'(ta + sb) - st(t'a' + s'b')] = 0$. Therefore since xy is in S, $(ta + sb)/(st) = (t'a' + s'b')/(t's')$.

The ring $S^{-1}R$ will consist of just one element if the 0 element is in S. For if r/s is in $S^{-1}R$, then $r/s = 0$, since $0(sr - 0) = 0$. It is for this reason that 0 cannot be in S. When S is the complement of a prime

ideal P, the ring of quotients will be denoted by R_P instead of $S^{-1}R$. In the case where S is the set of all non-zero divisors of R, then $S^{-1}R$ will be called the classical ring of quotients of R and will be denoted by $S_c^{-1}R$. If R is an integral domain and S is the set of all non-zero elements of R, then $S^{-1}R$ is a field, called the quotient field of R. For example if R is the ring of integers, then its quotient field is the field of rationals. Every element of S is invertible in $S^{-1}R$, for if s is in S, then $1/s = s^{-1}$ is in $S^{-1}R$.

There is a natural ring homomorphism ∂ from R into $S^{-1}R$ defined by $a\partial = a/1$. This is a homomorphism since if a and b are in R, then $(a+b)\partial = (a+b)/1 = a/1+b/1 = a\partial + b\partial$, and $(ab)\partial = (ab)/1 = (a/1)(b/1) = (a\partial)(b\partial)$. The kernel of ∂ is the set of elements annihilated by some member of S. That is, if x is in the kernel of ∂ then there exists an s in S such that $xs = 0$. The natural homomorphism ∂ is not, in general, injective (one-to-one). For example, suppose R is the ring $Z/(12)$, and $S = R-(\overline{2})$. That is, $S = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}, \overline{9}, \overline{11}\}$. Then $\overline{4}$ is in the kernel of ∂ since $\overline{3}\cdot\overline{4} = 0$. If, however, R is an integral domain, then ∂ is injective. For if r is an element of the kernel of ∂, then $r\partial = r/1 = 0/s$. Thus there exists a t in S such that $tsr = 0$. Since ts is in S and R is an integral domain, then $r = 0$. Therefore ∂ is one-to-one.

There is a correspondence between ideals in R and ideals in $S^{-1}R$.

Theorem 1. If A is a subset of R and $\bar{A}$ a subset of $S^{-1}R$ and $A\partial = \bar{A}$, then $\bar{A}\partial^{-1}$ is an ideal if and only if $\bar{A}$ is an ideal. Further, $\bar{A} = \{a/s \mid a \text{ in } A,\ s \text{ in } S\}$.

Proof: Let $\bar{A}\partial^{-1}$ be an ideal in R. Then ar and a−b are in $\bar{A}\partial^{-1}$ for any a and b in $\bar{A}\partial^{-1}$ and r in R. Thus $(a/1)(r/1) = (ar)/1 = (ar)\partial$ is in $\bar{A}$, and $(a/1 - b/1) = (a-b)/1 = (a-b)\partial$ is in $\bar{A}$. Therefore $(a/1)(r/1)$ and $a/1 - b/1$ are in $\bar{A}$ for $a/1$, $b/1$ in $\bar{A}$ and $r/1$ in $S^{-1}R$, so $\bar{A}$ is an ideal. Now suppose $\bar{A}$ is an ideal in $S^{-1}R$, then $(a/1)(r/1)$ and $a/1 - b/1$ are in $\bar{A}$ for any $a/1$, $b/1$ in A and $r/1$ in $S^{-1}R$. But $(a/1)(r/1) = (ar)/1$ in $\bar{A}$, thus ar is in $\bar{A}\partial^{-1}$. Also $a/1 - b/1 = (a-b)/1$ is in $\bar{A}$, thus $(a-b)$ is in $\bar{A}\partial^{-1}$. Therefore ar and a−b are in $\bar{A}\partial^{-1}$ for some a and b in $\bar{A}\partial^{-1}$ and r in R, so $\bar{A}\partial^{-1}$ is an ideal. Now let $M = \{a/s \mid a \text{ in } A,\ s \text{ in } S\}$, and let $a/1$ be in $\bar{A}$. Then $a/1$ is obviously in M since 1 is in S. Thus $\bar{A} \subset M$. Now let a/s be in M, then $a/s = (a/1)(1/s)$ is in $\bar{A}$ since $\bar{A}$ is an ideal and $a/1$ is in $\bar{A}$ and $1/s$ is in $S^{-1}R$. Thus $M \subset \bar{A}$. Therefore $\bar{A} = \{a/s \mid a \text{ in } A,\ s \text{ in } S\}$.

The ideal $\bar{A}$ explodes to $S^{-1}R$ if and only if A contains an element of S. For if $\bar{A} = S^{-1}R$, then 1 is in $\bar{A}$. So if s is in $S \subset R$, then $s \cdot 1 = s$ is in A since A is an ideal. Thus $A \cap S \neq \emptyset$. Now if a is in $A \cap S$, then $a/a = 1$ is in $\bar{A}$. Therefore $\bar{A} = S^{-1}R$. The correspondence between ordinary ideals in R and $S^{-1}R$ is not necessarily order preserving. That is, if A and B are ideals in R and $A \subset B$, then $\bar{A}$ might equal $\bar{B}$ in

$S^{-1}R$. Also on returning to R, $\bar{A}\phi^{-1}$ might possibly be the whole ring R. The correspondence improves if only prime ideals are considered as the following theorem shows.

Theorem 2. The natural homomorphism implements a one-to-one order preserving correspondence between all prime ideals in $S^{-1}R$ and those prime ideals in R disjoint from S.

Proof: Let P be a prime ideal in R such that $P \cap S = \emptyset$. By theorem 1 of this chapter $\bar{P}$ is an ideal in $S^{-1}R$ since $\bar{P} = P\phi$. The ideal $\bar{P}$ is proper since $P \cap S = \emptyset$. The ideal $\bar{P}$ is prime since if $(r/s)(a/b) = (ra)/(sb)$ is in P, then $(ra)/(sb) = p/s'$ for some p in P, s' in S. Thus there exists an s'' in S such that $s''s'ra = s''sbp$. Now s''sbp is in P, so s''s'ra is in P. Thus r is in P or a is in P since P is prime, and s's'' is not in P. Therefore r/s or a/b is in $\bar{P}$, and so $\bar{P}$ is prime. To show ϕ is one-to-one, let P and Q be prime ideals in R, and assume $\bar{P} = \bar{Q}$. Suppose p is in P, and let s be in S. Then p/s is in $\bar{P}$. Since $\bar{P} = \bar{Q}$, then $p/s = q/t$ in $\bar{Q}$ for some q in Q and t in S. Thus there exists an s' in S such that $s'tp = s'sq$. Therefore p is in Q since Q is a prime ideal and $S \cap Q = \emptyset$. Now suppose q is in Q and let s be in S. Then q/s is in $\bar{Q}$. Thus $q/s = p/t$ in $\bar{P}$ for some p in P, t in S, and there exists an s'' in S such that $s''tq = s''st$. Therefore q is in P since P is a prime ideal and $S \cap P = \emptyset$. Therefore $P = Q$. Since ϕ is one-to-one, then $\bar{P}\phi^{-1} = P$. That is, $\bar{P}$ returns to P. Thus if $\bar{P} \subseteq \bar{Q}$, then

$\overline{P}\varphi^{-1} \subseteq \overline{Q}\varphi^{-1}$ and so $P \subseteq Q$. For if $P \not\subseteq Q$, then there exists an element p in P such that p is not in Q. Thus p/s is not in $\overline{Q}$ and so $\overline{P} \not\subseteq \overline{Q}$. Therefore order is preserved.

Thus the maximal ideals in $S^{-1}R$ are simply the maximal prime ideals disjoint from S, which were discussed in the previous chapter. If we apply theorem 2 to the case where S is the complement of a prime ideal P, then theorem 2 implies that R_P has exactly one maximal ideal; thus R_P is a local ring. (A local ring is defined to be a commutative ring with a unique maximal ideal.) To see that R_P is a local ring, note that if r/s is not in $\overline{P}$, then r is not in P. Thus r is in S and so r/s is a unit in R_P. It follows that if $\overline{A}$ is an ideal in R_P and $\overline{A} \supset \overline{P}$, then $\overline{A}$ contains a unit and $\overline{A} = R_P$. Therefore $\overline{P}$ is the unique maximal in R_P. The necessary condition that $S^{-1}R$ be a local ring is given in theorem 3 that follows.

Theorem 3. Let T be a localization of R and assume that T is local. Then T has the form R_P with P a prime ideal in R.

Proof: Since T is a localization of R, then there is a multiplicative set S in R such that $T = \{a/s \mid a \text{ is in } R,\ s \text{ in } S\}$. Define $P = S'$ the complement of S. If a and b are in P, then a and b are not in S. Thus a/1, b/1 are not units in T. Since T is local a/1 and b/1 are in $\overline{M}$, the maximal ideal of T. Thus (a − b)/1 is in $\overline{M}$, so is not a unit in T. Therefore a − b is in P. Now if a is in P, then a/1 is in $\overline{M}$ in T,

and $(a/1)(r/s)$ is in $\overline{M}$ for all r in R, s in S. Thus $(ra)/s$ is not a unit in T, and ra is in P. Therefore P is an ideal. To show P is prime assume ab is in P. Thus ab is not in S so $(ab)/s$ is not a unit in T. Thus $(ab)/s$ is in $\overline{M}$. We may consider $(ab)/s$ as $(ab)/1 \cdot 1/s$ or $a/s \cdot b/1$ or $a/1 \cdot b/s$. For example, $(ab)/s = (ab)/1 \cdot 1/s$. Since $\overline{M}$ is prime in T, either $(ab)/1$ is in $\overline{M}$ or $1/s$ is in $\overline{M}$. Since s is a unit in T, $1/s$ is not in $\overline{M}$, thus $(ab)/1$ is in $\overline{M}$. But $(ab)/1 = (a/1)(b/1)$, so either $a/1$ or $b/1$ is in $\overline{M}$ since $\overline{M}$ is prime. If $a/1$ is in $\overline{M}$, then a is not in S. Thus a is in P. Similarly for $b/1$. Therefore either a or b is in P, and P is prime.

Not only is there a correspondence between ideals in R that are disjoint from S and ideals in $S^{-1}R$, but also some of the properties that the ideals may have in R are preserved in $S^{-1}R$. Examples include the following. In each case $\overline{A} = Ac$ for some ideal A in R.

1. If A is a principal ideal in R, then $\overline{A}$ is a principal ideal in $S^{-1}R$. Since $A = (a)$, then $Ac = (a)c = (a/1) = \overline{A}$. Therefore $\overline{A}$ is principal.

2. If A has a primary decomposition in R, then $\overline{A}$ has a primary decomposition in $S^{-1}R$. (An ideal A is said to have a primary decomposition if A can be expressed in the form $A = Q_1 \cap Q_2 \cap Q_3 \cap \ldots \cap Q_n$, where each Q_i is primary.) The proof that $\overline{A}$ has a primary decomposition can be found in Northcutt [10, p. 18].

3. If Q is a P-primary ideal in R, then $\overline{Q}$ is a $\overline{P}$-primary ideal in $S^{-1}R$. (An ideal Q is called a P-primary ideal if for ab in Q, with a not in P, then b is in Q.) For, let $x = q/s$, $y = q'/s'$ where x and y are elements of $S^{-1}R$ with the properties that $xy = (qq')/(ss')$ is in $\overline{Q}$ and $x = q/s$ is not in $\overline{P}$. Then qq' is in Q, while q is not in P. Thus q' is in Q. Thus $y = q'/s'$ is in $\overline{Q}$, and so $\overline{Q}$ is a $\overline{P}$-primary ideal.

4. If A is invertible in an integral domain R, then $\overline{A}$ is invertible in $S^{-1}R$. (The inverse of A, denoted A^{-1}, is defined to be the set of all x in the quotient field of R with the property that $xA \subseteq R$. An ideal A is invertible if $AA^{-1} = R$.) Suppose that $AA^{-1} \neq S^{-1}R$. Then $(AA^{-1})\mathfrak{c}^{-1} \neq (S^{-1}R)\mathfrak{c}^{-1}$ since R is an integral domain. Thus $(A\mathfrak{c}^{-1})(A^{-1}\mathfrak{c}^{-1}) \neq (S^{-1}R)\mathfrak{c}^{-1}$. Therefore $AA^{-1} \neq R$.

5. If A is a dense ideal in R, then $\overline{A}$ is a dense ideal in $S^{-1}R$ where S is a set of non-zero divisors in R. (An ideal A is dense if for all r in R, $rA = 0$ implies $r = 0$.) Suppose $\overline{A}$ is not dense in $S^{-1}R$ where $A\mathfrak{c} = \overline{A}$. Then there exists an r/s in $S^{-1}R$ such that $(r/s)\overline{A} = 0$ and $r/s \neq 0$. That is, for every a/t in A, $(r/s)(a/t) = (ra)/(at) = 0$. Thus $ra = 0$ for every a in A, so $rA = 0$. But A is dense in R, so $r = 0$, and thus $r/s = 0$, a contradiction. Therefore $\overline{A}$ is a dense ideal in $S^{-1}R$.

6. If A is the nilradical of R, then $\overline{A}$ is the nilradical of $S^{-1}R$. (The nilradical of R is the intersection of all prime ideals in R.) Since A is the nilradical of R, then $A = \cap P_i$ where each P_i is a prime

ideal in R. Let r/s be an element in $\bar{A}$ with r in A. Then r is in P_i for every P_i in R. Thus $r/1 = r/1$ is in $\bar{P}_i$ for every $\bar{P}_i$ in $S^{-1}R$. Therefore $(r/1)(1/s) = r/s$ is in $\bar{P}_i$ for every $\bar{P}_i$ and so r/s is in $\cap \bar{P}_i$. Now let p/s be in $\cap \bar{P}_i$. Then p/s is in $\bar{P}_i$ for every $\bar{P}_i$ in $S^{-1}R$. Thus p is in P_i for every P_i in R and so p is in A. Therefore p/s is in $\bar{A}$.

7. If P is a prime ideal in R that is disjoint from S, then the rank of P = the rank of $\bar{P}$ in $S^{-1}R$. (We say that P has rank n if there exists a chain of distinct prime ideals of length n descending from P, but no longer chain exists.) If we let P be of rank n, then $P = P_0 \supset P_1 \supset \ldots \supset P_n$, where each P_i is prime. Since there is a one-to-one, order preserving correspondence between prime ideals in R that are disjoint from S and prime ideals in $S^{-1}R$, then $\bar{P} = \bar{P}_0 \supset \bar{P}_1 \supset \ldots \supset \bar{P}_n$. Thus $\bar{P}$ is of rank n also.

The following theorem shows that finite sums, products, and intersections of ideals in R are preserved in $S^{-1}R$.

Theorem 4. Let S be a multiplicative set in R. Then for ideals I and J in R:

$$S^{-1}(I+J) = S^{-1}I + S^{-1}J$$

$$S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$$

$$S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J.$$

Proof: First we will show that $S^{-1}(I+J) = S^{-1}I + S^{-1}J$. If we let

x/s be an element of $S^{-1}(I+J)$, then x is in $(I+J)$, so $x = i+j$ for some i in I, and j in J. Thus $x/s = (i+j)/s = i/s + j/s$. Therefore x/s is in $S^{-1}I+S^{-1}J$. Now let y be an element of $S^{-1}I+S^{-1}J$. Then $y = i/s+j/s' = (s'i+sj)/ss'$ where $s'i$ is in I, sj is in J. Thus $s'i+sj$ is in $I+J$. Therefore y is in $S^{-1}(I+J)$. To show that $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$, we let x/s be an element in $S^{-1}(IJ)$. Then x is in IJ and $x = \sum_{i=1}^{n} a_i b_i$ for some a_i in I, b_i in J. Thus $x/s = \sum_{i=1}^{n}(a_i b_i)/s = \sum_{i=1}^{n}(a_i/s)(b_i/1)$ in $(S^{-1}I)(S^{-1}J)$. Now let z be an element in $(S^{-1}I)(S^{-1}J)$. Then $z = \sum_{i=1}^{n}(x/s)_i\ (y/s')_i = \sum_{i=1}^{n}(x_i y_i)/(s_i s_i')$ for some x_i/s_i in $S^{-1}I$, y_i/s' in $S^{-1}J$. Thus $z = (x_1 y_1)/(s_1 s_1')+(x_2 y_2)/s_2 s_2')+ \ldots +(x_n y_n)/s_n s_n') =$

$(s_2 s_2' \ldots s_n s_n' x_1 y_1 + \ldots + s_1 s_1' \ldots s_{n-1} s_{n-1}' x_n y_n)/(s_1 s_1' s_2 s_2' \ldots s_n s_n') =$

$(x_1' y_1 + x_2' y_2 + \ldots + x_n' y_n)/s = \sum_{i=1}^{n}(x_i' y_i)/s$ in $S^{-1}(IJ)$ where $x_i' = \prod_{\substack{j=1 \\ j\neq i}}^{n} s_j s_j' x_i$. Therefore z is in $S^{-1}(IJ)$. Now let x/s be an element in $S^{-1}(I \cap J)$. Then x is in $(I \cap J)$ so x is in both I and J. Thus x/s is in $S^{-1}I$ and $S^{-1}J$. Therefore x/s is in $S^{-1}I \cap S^{-1}J$. Now if y/s is in $S^{-1}I \cap S^{-1}J$, then y/s is in both $S^{-1}I$ and $S^{-1}J$. Thus y is in I and J. Therefore y is in $I \cap J$ and so y/s is in $S^{-1}(I \cap J)$.

It might be noted here that the passage from R to $S^{-1}R$ cuts out all prime ideals except those contained in P, where P is maximal with respect to the exclusion of S. That is, there are no proper prime ideals in $S^{-1}R$ that contains P. For if a prime ideal I contains P and P is maximal with respect to the exclusion of S, then I contains an

element of S and so $\bar{I} = S^{-1}R$. The passage from R to R/P, the residue ring modulo P, cuts out all prime ideals except those containing P. Thus if we localize the ring R with respect to P and then obtain the residue ring modulo $\bar{P}$ the result is a field.

Theorem 5. Let P be a prime ideal. The quotient field of R/P is isomorphic to $R_P/\bar{P}$ where $\bar{P}$ is prime in R_P.

The proof of theorem 5 is not difficult, but it is long and tedious so it will be omitted. The proof can be found in Barshay [2, p. 35].

The number of prime ideals in a ring indicates to some degree the number of localizations a ring might have.

Theorem.6. A ring R has an infinite number of localizations if and only if it has an infinite number of prime ideals.

Proof: Suppose R has an infinite number of localizations. Thus there exist multiplicative sets $S_1, S_2, \ldots$ such that $S_1^{-1}R \neq S_2^{-1}R \neq \ldots$. Let P_i be an ideal in R maximal with respect to the exclusion of S_i. Thus P_i is prime. Since $P_i \subseteq S_i'$, $P_j \subseteq S_j'$, where S_i' is the complement of S_i, then $P_i \neq P_j$. For if $P_i = P_j$ then $\bar{P}_i = \bar{P}_j$ where $\bar{P}_i = P_i\partial$. Thus $\bar{P}_i' = \bar{P}_j'$. But $\bar{P}_i' = S_i^{-1}R - \bar{P}_i$ and $\bar{P}_j' = S_j^{-1}R - \bar{P}_j$. Thus $S_i^{-1}R - \bar{P}_i = S_j^{-1}R - \bar{P}_j = S_j^{-1}R - \bar{P}_i$, and so $S_i^{-1}R = S_j^{-1}R$. A contradiction, thus $\bar{P}_i \neq \bar{P}_j$, and R has an infinite number of prime ideals. Now suppose R has an infinite number of prime ideals. Then define $S_i = P_i'$ and thus $S_1 \neq S_2 \neq \ldots$. Therefore $S_1^{-1}R \neq S_2^{-1}R \neq \ldots$, and R has an

infinite number of localizations.

An example of a ring with an infinite number of localizations is the ring of integers J, for $J_{(2)} \neq J_{(3)} \neq \ldots$. Notice that in this case we have $J \subset J_{(2)} \subset Q$, where Q is the field of rationals. For $J \subset J_{(2)}$ since if x is in J, then $x = x \cdot 1^{-1} = x/1$ in $J_{(2)}$. Also $J \subset J_{(3)} \subset Q$. In fact every ring between the ring of integers J and the field of rationals Q is a ring of quotients of J as shown in the next theorem.

Theorem 7. Every ring between the ring of integers and the field of rationals is a ring of quotients of J where J is the ring of integers and Q is the ring of rationals.

Proof: Let R be a ring such that $J \subseteq R \subseteq Q$ and let $S = \{m \text{ in } J \mid n/m \text{ is in } R \text{ for some } n, (n, m) = 1\}$. The set S is a multiplicative set, since if a and b are in S then $(n_1, a) = 1$ and $(n_2, b) = 1$ for some n_1, n_2 in J. Thus there exist integers x, y, s, and t in J such that $xa + yn_1 = 1$ and $sb + tn_2 = 1$. Hence $1/a = y(n_1/a) + x$ is in R since $y(n_1/a)$ is in R and x is in $J \subseteq R$. Similarly 1/b is in R, and so $(1/a)(1/b)$ is in R. Therefore ab is in S, and so $R = S^{-1}J$ is a ring of quotients of J.

The field of quotients of J is also the field of quotients of $J_{(2)}$, $J_{(3)}$, . . . by the following theorem.

Theorem 8. Let R be a ring and let S be a multiplicative set of R which contains no zero divisors. If R_1 contains R and is contained in

$S^{-1}R$, then $S^{-1}R_1 = S^{-1}R$.

Proof: Since $R \subseteq R_1$, then $S^{-1}R \subseteq S^{-1}R_1$ obviously. Now let r'/s be in $S^{-1}R_1$, where r' is in R_1 and s is in $S \subset R$. Then $r' = r/s_1$, with r in R, s_1 in S, since $R_1 \subseteq S^{-1}R$. Thus $r'/s = r/ss_1$, so r'/s is in $S^{-1}R$. Therefore $S^{-1}R = S^{-1}R_1$.

Corollary. Let R be an integral domain and let F be the quotient field of R. If R_1 is an integral domain and $R \subset R_1 \subset F$, then F is the quotient field of R_1.

Proof: Let S and S_1 be the set of non-zero elements of R and R_1 respectively. Obviously $S \subseteq S_1$, so let s be in S_1. Then $1/s$ is in $S^{-1}R$, since $S_1 \subset R_1 \subset S^{-1}R$. Therefore $S = S_1$. Thus $S^{-1}R_1 = S_1^{-1}R_1$. Now $S^{-1}R = S^{-1}R_1$ by theorem 8, and $F = S^{-1}R$, so $F = S_1^{-1}R_1$, the quotient field of R_1.

Since in the example $J \subset J_{(2)}$, $J \subset J_{(3)}$, . . . , then we have $J \subset \bigcap J_{P_i}$. Actually the stronger result that $J = \bigcap J_{P_i}$ holds.

Theorem 9. Let R be any integral domain. Then $R = \bigcap R_m$, the intersection ranging over the maximal prime ideals in R.

Proof: Obviously $R \subseteq R_m$ for each R_m and therefore $R \subseteq \bigcap R_m$. Let x be in $\bigcap R_m$. Thus $x = r/s$ where r and s are in R, and s is not in any maximal ideal M of R. Consider the ideal D of R consisting of all elements y in R such that xy is in R. Since $x = r/s$, then $sx = r$ is in R. Hence s is in D, and so D is not a subset of any maximal ideal in R. Hence $D = R$. Therefore 1 is in D and so $1 \cdot x = x$ is in R.

Therefore if an integral domain R has only one non-zero prime ideal, then from theorem 9 $R = R_P$, and this is the case when $P \neq R$.

Theorem 10. If R is an integral domain with quotient field K, and $R \neq K$, then R has exactly one non-zero prime ideal if and only if the only localizations of R are R and K.

Proof: Suppose R has exactly one non-zero prime ideal, say P. Since $P \neq R$, then R is local and $S^{-1}R = \{r/s \mid r \text{ is in } R,\ s \text{ in } S\}$, where S is such that P is maximal with respect to the exclusion of S. Obviously $R \subseteq S^{-1}R$. So let r/s be in $S^{-1}R$. Then s is not in P. Thus s is a unit in R since R is local, and so r/s is in R. Therefore $R = S^{-1}R$. Now suppose R has two distinct non-zero prime ideals P_1 and P_2. Then $R_{P_1} \neq R_{P_2}$ and both are different from K since K is a localization of R with respect to the prime ideal (0), a contradiction.

We observe that we have established in the proof above that R is its own ring of quotients; that is, $R = S^{-1}R$, when $P \neq R$ in an integral domain R if P is the only non-zero prime ideal. The next theorem shows that there are other times when the ring of quotients $S^{-1}R$ equals R. We recall first that an element u in T is said to be integral over R if it satisfies an equation of the form $u^n + a_1 u^{n-1} + \ldots + a_n = 0$, with all a_i in R, where T is a ring containing R. We say that T is integral if all its elements are integral. The elements of T that are integral over R form a subring of T [8, p. 10], called the integral closure of R in T.

Theorem 11. Let R be an integral domain. If a localization $S^{-1}R$ is integral over R, then $R = S^{-1}R$.

Proof: Obviously $R \subseteq S^{-1}R$. Now if $S^{-1}R$ is integral over R then for every s^{-1} in $S^{-1}R$, s^{-1} is integral over R. Also since every s in S is invertible in $S^{-1}R$ and s^{-1} is integral over R, then s^{-1} is in R[s] by a result of Kaplansky [8, p. 10]. Since s is in R, then R[s] = R, and so s^{-1} is in R. Therefore $R = S^{-1}R$.

There are other examples when $R = S^{-1}R$ in the next chapter where the classical rings of quotients are considered. Before proceeding to the special case we should consider, at this point, how integral closure behaves relative to localization. An integral domain is said to be integrally closed if every x in K, the quotient field of R, which is integral over R is in R.

Theorem 12. If R is an integrally closed integral domain and if S is a multiplicative set in R, then $S^{-1}R$ is integrally closed.

Proof: Suppose that the element u is in the quotient field and is integral over $S^{-1}R$. We are given say, $u^n + (a_1/s_1)u^{n-1} + \ldots + (a_n/s_n) = 0$ with a_i in R, s_i in S. Put $s = s_1 s_2 \ldots s_n$ and $t_i = s/s_i$. Then $su^n + t_1 a_1 u^{n-1} + \ldots + t_n a_n = 0$. If we multiply by s^{n-1} we get an equation asserting that su is integral over R. Hence su is in R, so $u = (su)/s$ is in $S^{-1}R$.

Theorem 13. If R is an integral domain with integral closure T,

then the integral closure of $S^{-1}R$ is $S^{-1}T$.

Proof: If u is in K, the quotient field of R, and u is integral over $S^{-1}R$, then $u^n + a_1/s_1 u^{n-1} + \ldots + a_n/s_n = 0$ with a_i in R, s_i in S. Let $s = s_1 s_2 \ldots s_n$ and $t_i = s/s_i$. Then $su^n + t_1 a_1 u^{n-1} + \ldots + t_n a_n = 0$ and $s^n u^n + a_1 t_1 s^{n-1} u^{n-1} + \ldots + a_n t_n s^{n-1} = 0 = (su)^n + a_1 t_1 (su)^{n-1} + \ldots + a_n t_n s^{n-1} s^0 u^0$. Hence su is integral over R. So $u = (su)/s$ is in $S^{-1}T$. Now let u be in $S^{-1}T$. Then $u = t/s$ for some t in T, s in S. Since T is integral over R, then $t^n + a_1 t^{n-1} + \ldots + a_n = 0$. Thus $(t^n + a_1 t^{n-1} + \ldots + a_n)/s^n = 0/s^n$ in $S^{-1}T$. Hence $(t/s)^n + (a_1/s)(t/s)^{n-1} + \ldots + a_n/s^n = 0$. Therefore u is integral over $S^{-1}R$, and so is in the integral closure of $S^{-1}R$.

CHAPTER III

CLASSICAL RINGS OF QUOTIENTS

In this chapter we will examine some of the properties of the classical ring of quotients, denoted $S_o^{-1}R$, where R is any commutative ring with an identity element and at least one non-zero divisor and S_o is the set of non-zero divisors in R. This ring is called the classical ring of quotients since it was the first such quotient ring studied (by Grell). As we shall see, the classical ring of quotients possesses many nice properties that an ordinary ring of quotients may not necessarily possess. Since the classical ring of quotients is just a special case of the rings of quotients, all the results shown in the last chapter hold. One nice property the set S_o has is stated in the following theorem.

Theorem 1. The set S_o is the largest multiplicative set S for which the natural homomorphism $\varphi: R \to S^{-1}R$ is injective.

Proof: This theorem will be proved by showing that φ is injective if and only if S contains no zero divisors. So suppose φ is injective and there exists an s in S such that $s \neq 0$ and s is a zero divisor. The $sr = 0$ for some $r \neq 0$ in R. Thus $(sr)/s = 0/s$ and $r/s = 0/s$ in $S^{-1}R$. Hence φ is not one-to-one. Now suppose S contains no zero divisors, and let r be in the kernel of φ. Then $r\varphi = r/s = 0/s$.

Thus there exists an s' in S such that $s'sr = 0$, and $r = 0$ since S contains no zero divisors. Thus ϕ is injective. Now since S_0 contains all the non-zero divisors in R, then any set S_1 containing S_0 would necessarily contain a zero divisor. Therefore ϕ would not be injective for S_1.

Any commutative ring R with at least one non-zero divisor possesses a classical ring of quotients. Since the proof that R possesses a classical ring of quotients is long, it will be omitted. The proof can, however, be found in Samuel [13, p. 44].

When the ring R is an integral domain, then $S_0 = R - \{0\}$. Thus $S_0^{-1}R$ forms a field, called the field of quotients. The field of quotients constructed from the ring of integers is, of course, the field of rationals, and this field is the smallest field which contains the integers. This is a special case of the more general result:

Theorem 2. Any field F containing an integral domain R as a subring contains the field of quotients $S_0^{-1}R$.

Proof: Let x/s be in $S_0^{-1}R$ for some x in R, s in $S_0 = R - \{0\}$. Since s is in R and $R \subset F$ then s is in F. Thus $s^{-1} = 1/s$ is in F. Therefore x/s is in F and so $S_0^{-1}R \subset F$.

That any two quotient fields of an integral domain R are isomorphic is a consequence of the next theorem, whose proof is straightforward but tedious and can be found in Samuel [13, p. 43].

Theorem 3. Let R and R_1 be two commutative rings each containing at least one non-zero divisor. Then, any isomorphism of R onto R_1 has a unique extension to an isomorphism of $S_o^{-1}R$ onto $S_o^{-1}R_1$.

If the set S contains only non-zero divisors in a ring R, but not necessarily all the non-zero divisors of R, then $S^{-1}R$ is a subring of $S_o^{-1}R$.

Theorem 4. Let S be any multiplicative set of the ring R which contains no zero divisors of R. If the set T is defined by $T = \{a/s \text{ in } S_o^{-1}R \mid a \text{ is in } R,\ s \text{ in } S\}$ then T is a subring of $S_o^{-1}R$.

Proof: Obviously $T \subseteq S_o^{-1}R$, so we need only prove that T is a ring. Let a/s and b/t be in T. Then $a/s - b/t = (ta - sb)/st$ where st is in S since S is a multiplicative set. Also st is in S_o since st is not a zero divisor, and $ta - sb$ is in R since R is a ring and ta and sb are in R. Thus $a/s - b/t$ is in T. Now $(a/s)(b/t) = (ab)/(st)$ is in T since st is a non-zero divisor and ab is in R. Therefore T is a subring of $S_o^{-1}R$.

That it is possible to extend the previous theorem to a chain of subrings of $S_o^{-1}R$ has been shown in Gilmer [6, p. 15] in his theorem 1.6.

Theorem 5. Let T be the classical ring of quotients of the ring R and let $R \subseteq M \subseteq M_1 \subseteq T$ where M and M_1 are rings.

a) If M_1 is a ring of quotients of R, then M_1 is a ring of quotients of M.

b) If M is a ring of quotients of R and M_1 is a ring of quotients of M, then M_1 is a ring of quotients of R.

As in the general case, if $S_o^{-1}R$ is the classical ring of quotients of R and if R_1 is an intermediate ring, then theorem 5 shows that $S_o^{-1}R$ is a ring of quotients of R_1. It turns out that $S_o^{-1}R$ is also the classical ring of quotients of R_1. For if x is in R_1, then x is of the form r/s for some r and s in R with s a non-zero divisor. Further it can easily be seen that if x is a non-zero divisor in R_1, then r must be a non-zero divisor in R, since an element r/s is a non-zero divisor of $S_o^{-1}R$ if and only if r is a non-zero divisor of R. Consequently, x is a unit of $S_o^{-1}R$. Thus if $S_o^{-1}R$ is the classical ring of quotients of R and if R_1 is an intermediate ring, then $S_o^{-1}R = S_o^{-1}R_1$. In particular, $S_o^{-1}R$ is a classical ring of quotients of $S_o^{-1}R$ and therefore each non-zero divisor of $S_o^{-1}R$ is a unit in $S_o^{-1}R$.

Theorem 6. Every element of $S_o^{-1}R$ is either a zero divisor or a unit.

Proof: Let r/s be an element of $S_o^{-1}R$ and suppose r/s is a non-zero divisor in $S_o^{-1}R$. Hence r is a non-zero divisor in R, and so r is in S_o. Thus r/s is a unit in $S_o^{-1}R$.

We shall now look at some more examples of rings that are identical with their classical ring of quotients.

1. If R has an identity and every non-zero divisor of R is

invertible, then $R = S_o^{-1}R$. Obviously $R \subseteq S_o^{-1}R$, so let r/s be in $S_o^{-1}R$, for r in R and s in S. Then s is invertible in R, thus $r/s = rs^{-1}$ is in R. Therefore $S_o^{-1}R \subseteq R$, and so $R = S_o^{-1}R$.

2. For a commutative ring R with identity, $R = S_o^{-1}R$ if and only if every non-unit is a zero divisor. For suppose $R = S_o^{-1}R$ and s is a non-zero divisor in R. Then s is in S_o, so $1/s$ is in $S_o^{-1}R$. But $R = S_o^{-1}R$, so $1/s$ is in R. Thus s is a unit in R, proving that any non-unit in R is a zero-divisor. The proof of the converse was given in example 1.

3. If R contains no non-zero divisors except the identity, then $R = S_o^{-1}R$. For if r/s is in $S_o^{-1}R$, then $r/s = r/1 = r \cdot 1^{-1} = r$ in R. Thus $S_o^{-1}R \subseteq R$. That $R \subseteq S_o^{-1}R$ is obvious. Therefore $R = S_o^{-1}R$.

CHAPTER IV

PROPERTIES PRESERVED UNDER LOCALIZATION

In chapter 3 it was shown that many of the properties concerning ideals are preserved under localization. Now we will look at some of the properties of the ring which are preserved under localization. In all that follows, S is an arbitrary multiplicative set of the ring considered.

Theorem 1. If R is an integral domain, then $S^{-1}R$ is an integral domain.

Proof: Suppose $a/s \cdot b/t = 0/s'$ with a/s, b/t in $S^{-1}R$. Then $s'ab = st \cdot 0 = 0$, so $ab = 0$ since s' is in S and $s' \neq 0$. Then $a = 0$ or $b = 0$, so that $a/s = 0$ or $b/t = 0$. Therefore $S^{-1}R$ is an integral domain.

Theorem 2. If R is a principal ideal domain, then $S^{-1}R$ is a principal ideal domain.

Proof: Let $\overline{A}$ be an ideal in $S^{-1}R$. Then $\overline{A} = Ac$ for some ideal A in R.. Since $A = (a)$, then $Ac = (a)c = (a/1) = \overline{A}$. Therefore $\overline{A}$ is principal, and so $S^{-1}R$ is a principal ideal domain.

Theorem 3. If R is a unique factorization domain, then $S^{-1}R$ is a unique factorization domain.

Proof: An integral domain R is a unique factorization domain

if every non-zero element is expressible uniquely as a product of prime elements. That this is equivalent to the definition that a ring is a unique factorization domain if and only if every non-zero prime ideal in R contains a non-zero prime ideal which is principal can be found in Kaplansky [8, th. 5, p. 4]. Suppose $\overline{P}$ is a prime ideal in $S^{-1}R$, where $\overline{P} = P\mathcal{O}$, for some prime ideal P in R. Since R is a unique factorization domain, P contains an ideal (p), where (p) is prime and principal in R. Then $(p)\mathcal{O}$ is prime and principal in $S^{-1}R$, and obviously $(p)\mathcal{O} \subset \overline{P}$. Therefore $S^{-1}R$ is a unique factorization domain.

Theorem 4. If R is a Noetherian ring, then $S^{-1}R$ is a Noetherian ring.

Proof: A ring R is called a Noetherian ring if every prime ideal in R is finitely generated. Suppose $\overline{P}$ is a prime ideal in $S^{-1}R$. Then $\overline{P} = P\mathcal{O}$ for some prime ideal P in R, by theorem 1 of chapter 2. But P is finitely generated in R, so $\overline{P}$ is finitely generated in $S^{-1}R$. Therefore $S^{-1}R$ is Noetherian.

Theorem 5. If R is a valuation ring, then $S^{-1}R$ is a valuation ring.

Proof: A commutative ring R is said to be a valuation ring if for any a and b in R either a divides b or b divides a. Suppose r/s and t/s' are in $S^{-1}R$ and r/s does not divide t/s'. Thus (rs')/(ss') does not divide (ts)/(ss'), and so rs' does not divide ts. Since R is a valuation ring, then ts divides rs'. Hence (ts)/(ss') divides (rs')/(ss'), and so

t/s' divides r/s. Therefore $S^{-1}R$ is a valuation ring.

Theorem 6. If R is a Dedekind ring, then $S^{-1}R$ is a Dedekind ring.

Proof: If R is an integral domain and every non-zero divisor of R is invertible then R is said to be a Dedekind ring. Since R is an integral domain, then $S^{-1}R$ is an integral domain and by example 4 of chapter 1 every non-zero ideal $\bar{A}$ in $S^{-1}R$ is invertible if A is invertible in R where $\bar{A} = A\phi$. Therefore $S^{-1}R$ is a Dedekind ring.

Theorem 7. If R is a Prüfer ring, then $S^{-1}R$ is a Prüfer ring.

Proof: A Prüfer ring is an integral domain in which every non-zero finitely generated ideal is invertible. The ring $S^{-1}R$ is an integral domain, by theorem 1. Also every non-zero finitely generated ideal is invertible in R, since R is Prüfer. Since $\bar{A} = A\phi$ for some ideal A in R, then $\bar{A}$ is finitely generated in $S^{-1}R$. Also by example 4 of chapter 1 $\bar{A}$ is invertible in $S^{-1}R$ since A is invertible in R. Therefore $S^{-1}R$ is a Prüfer ring.

Gilmer [6, p. 556] shows that $Z[\sqrt{-5}]$ is a Prüfer ring. Since every Dedekind ring is a Prüfer ring, this is also an example of a Dedekind ring.

Theorem 8. If R is a Bezout ring, then $S^{-1}R$ is a Bezout ring.

Proof: A Bezout ring is an integral domain in which every finitely generated ideal is principal. Let $\bar{A}$ be a finitely generated ideal

in $S^{-1}R$, where $\overline{A} = Ac$ for some finitely generated ideal A in R. Since R is a Bezout ring, then A is also principal. Thus $\overline{A}$ is principal in $S^{-1}R$ by theorem 2 of chapter 4. Therefore $S^{-1}R$ is a Bezout ring.

Obviously every principal ideal ring is a Bezout ring. The ring of entire functions, that is, functions of a complex variable that are differentiable is another example of a Bezout ring. It can easily be shown that $Z[\sqrt{-5}]$ is not a unique factorization domain and so not a principal ideal domain. Hence $Z[\sqrt{-5}]$ is an example of a Prüfer ring that is not a Bezout ring.

Theorem 9. If R is integrally closed, then $S^{-1}R$ is integrally closed.

Proof: The proof of this was given in chapter 2, page 23.

Theorem 10. If R is absolutely flat, then $S^{-1}R$ is absolutely flat.

Proof: A ring R is called absolutely flat if every principal ideal A is idempotent, that is, $A^2 = A$. Let $\overline{A}$ be an ideal in $S^{-1}R$. Then $\overline{A} = Ac$ for some ideal A in R. Let a/s be an element in $\overline{A}$. Then a is in $A = A^2$. Thus a/s is in $\overline{A}^2$, so $\overline{A} \subseteq \overline{A}^2$. The reverse inclusion is obvious. Therefore $S^{-1}R$ is absolutely flat.

Theorem 11. If R is normal, then $S^{-1}R$ is normal.

Proof: An integrally closed domain is called a normal ring. Thus by theorem 9, $S^{-1}R$ is normal.

There are many more properties that are preserved under

localization, but to show them all would go beyond the scope of this paper.

Although the converse of many of the theorems in this chapter is not valid, there are times when we can look at $S^{-1}R$ and determine the properties of R. Just two of the many examples of going from $S^{-1}R$ to R follows.

1. If the local ring R_P has no non-zero nilpotent elements, then R has no non-zero nilpotent elements. For suppose x is an element in R where $x \neq 0$ and $x^n = 0$. Then x is in $\cap P_i$ where P_i ranges over all prime ideals in R. Thus x/s is in $\cap \overline{P}_i$ where $\overline{P}_i$ ranges over all prime ideals in R_P. Thus $(x/s)^n = x^n/s^n = 0/s^n$ is in $\cap \overline{P}_i$. Therefore R_P has nilpotent elements not equal to zero.

2. Let R be a commutative ring such that R_P is a principal ideal domain for every prime ideal P of R. Then every non-zero prime ideal in R is maximal. The proof of this example can be found in Barshay [2, p. 98].

As we have just seen, many of the important properties of rings are preserved under localization; therefore we can often pass from the ring R to the ring $S^{-1}R$ without losing the properties of the original ring in the process. In proving many theorems, the modern trend in several areas of algebra is to pass to the ring of quotients to obtain easier and simpler proofs of theorems that would otherwise be

difficult and tedious to prove. Northcutt [10], Samuel [13], and Kaplansky [8], use the technique of passing to the ring of quotients to obtain simpler proofs throughout their books. An example of how this technique shortens proofs is seen in the theorem that if R is a unique factorization domain, then R[x] is a unique factorization domain. This theorem can be proven in just a few sentences by passing to the quotient field of R. This is done in Barshay [2, th. 4-6, p. 46]. Otherwise, this theorem would be long -- about one and one half pages in Burton [3, th. 7-11, p. 124], and by no means easy to follow.

Since obtaining a short, simple proof of a theorem is desirable in most branches of mathematics, the rings of quotients deserve to be studied and carefully analyzed. However, obtaining simple proofs is not the only application of rings of quotients. Examples of recent applications to algebraic geometry can be found in the papers of Chevalley [4] and Zariski [12]. Applications can also be found in Gilmer's Multiplicative Ideal Theory and in Nagata's Local Rings. To try to show all the applications of the rings of quotients would go beyond the scope of this paper, since it would require an extended analysis of such areas as algebraic geometry and local rings.

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